

Solvability for the mathematical model representing
motions of the elastic curves on the plane

平面上の弾性体の伸縮運動を表す数理モデルの
可解性について

by

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Abstract

It is known that a shape memory alloy ring hanging on pulleys rotates rapidly, when its part touches hot water. For this interesting phenomenon we can consider a system of the kinetic and heat equations for elastic materials. However, since on mathematical and numerical analysis to the model there exist many difficulties, we propose new mathematical models for the motion on the plane by simplification.

In Chapter 1, we first show previous results on shape memory alloys and a beam equation. They are precious basis to construct our models for the closed elastic curves. Next, we provide three models and mention about their feature and advantage.

In Chapter 2 to Chapter 4, we give results for the ordinary differential equation (ODE) model, and the partial differential equation (PDE) models having a viscosity term and without it. The most important features are that the strain is nonlinear and the stress function has a singularity. Thanks to this singularity, we could obtain the lower bounds of the strains and prove existence and uniqueness of solutions and weak solutions of the ODE and PDE models, respectively. Moreover, for the PDE model with the viscosity term, we prove existence of strong solutions. We also show existence of periodic solutions in time and some numerical results for the ODE and PDE models.

Finally, in Chapter 5, for each model we list future tasks related to large time behavior of solutions, development of numerical schemes, and generalization of the theorems discussed in this thesis.

Chapter 1

Introduction

1.1 Mathematical problems of the shape memory alloy rings

In our modern life we can find a lot of industrial products utilizing elasticity. Shape memory alloys are well-known as an elastic material having the property of restoring deformed objects to the original state at a certain temperature. By applying this property to products, we can contribute to improve our life circumstance. For this aim it is very important to construct a mathematical model representing dynamics of shape memory alloys, since we can attempt any ideas without real experiments. Here, we note that from previous physics experiments this property involves stress and strain as well as temperature changes. It is not easy to deal with this stress-strain relationship depending on temperature mathematically, since it is not a simple functional relationship as in Figures 1.1, 1.2 obtained from experiments (see [9]). Subsequently, we can say that it is a challenge to construct an appropriate representation mathematically.

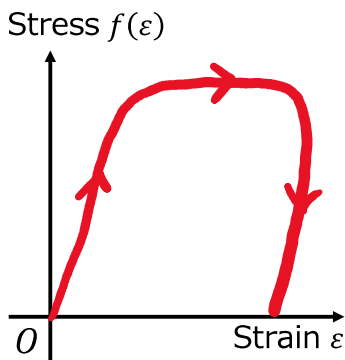


Figure 1.1: Image of relationship under lower temperature

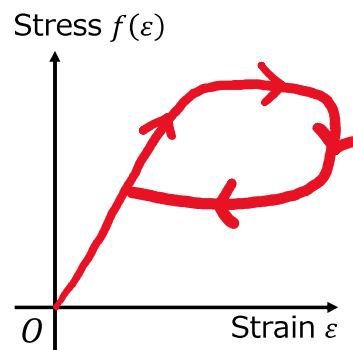


Figure 1.2: Image of relationship under higher temperature

This relationship which is not a function is called hysteresis, and has been researched, mathematically, even present day. For example, Brokate and Sprekels approximated the hysteresis by the polynomial in [9] for analysis to shape memory alloy. They assumed the domain of functions representing the phenomena is in one or three dimensional space. Here, we note that the domain and the range of functions are given in the same dimensional space. Other mathematical approach to hysteresis is application of subdifferential operators, for instance [3, 4, 2, 5]. However, these mathematical expressions have some issues with in terms of their physical background and mathematical treatment. In this research, therefore, we aim to give a new mathematical expression to the relationship between stress and strain with changing temperature. More precisely, the aim is to construct a mathematical model representing the rotational motion of shape memory alloy rings, partially submerged in hot water as in Figure 1.3. In this phenomena, the high speed rotating motion of shape memory alloy ring is observed, and it is caused by the temperature difference between the hot water and the air. This phenomenon is due to hysteresis and temperature dependence of elasticity.

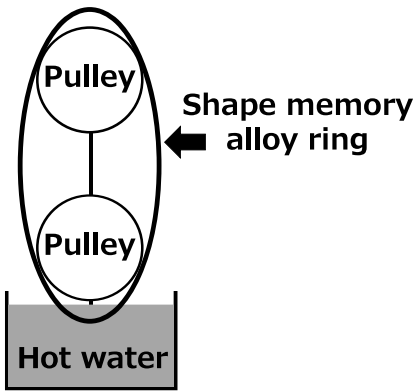


Figure 1.3: Rotational motion

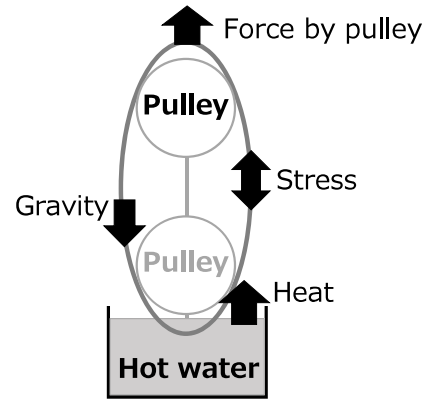


Figure 1.4: Modeling issues

In this modeling, we have a lot of difficulties. For example, we have to take into account of forces by pulleys, gravity, and stress of the shape memory alloy ring and heat flow (see Figure 1.4). However, for such a complex system, it is very difficult to construct a realistic and reasonable model and to handle it mathematically. Therefore, by simplifying the phenomenon, we focus on the elasticity of shape memory alloys, and construct a mathematical model representing motions of elastic curves such as rubber rings. In this chapter, we give several mathematical models obtained by some simplification and approximation. Before introducing them, in the next section we discuss about beam equations which play very important role in construction and investigation for the partial differential equation model.

1.2 Beam equation for representing motions of elastic materials

The equation (1.2.1) is called a beam equation, which is known as one of the partial differential equations for dynamics of beams (see[11, 27, 31, 32]).

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0 \text{ in } (0, T) \times (0, 1), \quad (1.2.1)$$

where $T > 0$. Also, the system of differential equations containing the beam equation was investigated well as a mathematical model for shape memory alloys in [9, 2]. In these results the unknown function represents the displacement, and the stress functions are supposed to be continuous on \mathbb{R} . For example, in [31, 32], Takeda and Yoshikawa consider the following semilinear beam equation with weak damping:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} - \alpha \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(f \left(\frac{\partial u}{\partial x} \right) \right), t > 0, x \in \mathbb{R}, \\ u(0, x) &= g_0(x), \frac{\partial}{\partial t} u(0, x) = g_1(x), x \in \mathbb{R}, \end{aligned}$$

where $f \in C^1(\mathbb{R})$ is the stress function, g_0 and g_1 are given initial functions, α is a positive constant. They have proved the existence and uniqueness of the mild solution for small initial data (g_0, g_1) . Also, the smoothing effect of the solution is established and the time decay estimates are obtained. Moreover, they give the asymptotic profiles of the solution.

Here, we note that, Okabe [26] studied elastic curves on the plane by the partial differential equations whose type is different from a beam equation.

Due to these previous studies, we construct the mathematical model representing stretching and shrinking motions of closed elastic curves on the plane with beam equations.

1.3 Outline of this thesis

This thesis consists of 5 chapters. In each chapter from Chapter 2 to Chapter 4, we state the results for the ordinary differential equation or partial differential equation models. These results are based on the following our published or submitted papers:

- The results of Chapter 2 is based on the joint-works with Toyohiko Aiki published in 2020 (cf. [6]) and Toyohiko Aiki, Martijn Anthonissen and Makoto Okumura published in 2021 (cf. [17]);
- the results of Chapter 3 is based on the joint-work with Toyohiko Aiki published in 2021 (cf. [7]);
- the results of Chapter 4 is based on my single authorship under peer review (cf. [18]).

We note that in Chapter 2, we improve the assumption of initial conditions of Theorems 2.1, 2.2 and 2.3 in [6]. Moreover, in Chapter 3, we add several lemmas and give those proofs, as well as the proofs of Lemmas 1 and 2 omitted in [7].

From now on, we give outlines of Chapters 2–4. In Chapter 2, we consider dynamics of the one-dimensional elastic material in \mathbb{R}^2 and assume that it is given by a polygon having N vertexes for each time t , and the natural length of each side is l_{N*} , where $N \in \mathbb{Z}_{>0} := \{n \in \mathbb{Z} \mid n > 0\}$. Let $X_{(i)} = X_{(i)}(t) \in \mathbb{R}^2$ be the position of each vertex of the polygon as in Figure 1.5. Hence, the strain $\varepsilon_{(i)}$ of i th side is given by

$$\varepsilon_{(i)} = \frac{l_{(i)} - l_{N*}}{l_{N*}}, \quad l_{(i)} = |X_{(i+1)} - X_{(i)}| \quad \text{for } i = 1, 2, \dots, N,$$

where $X_{(N+1)} = X_{(1)}$.

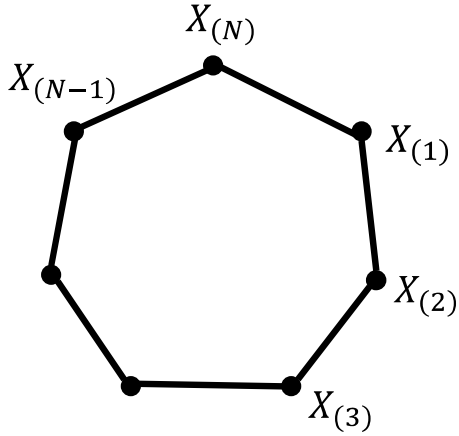


Figure 1.5

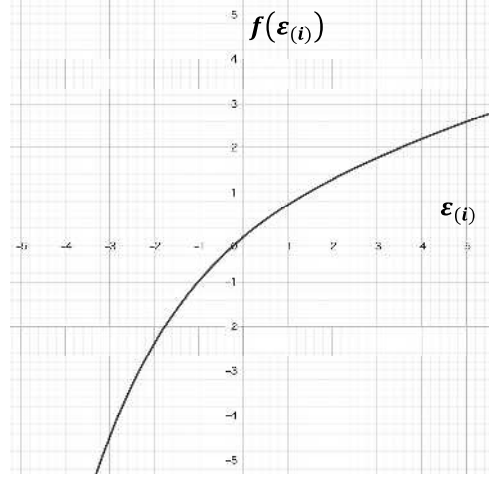


Figure 1.6

Next, we list our physical assumption as follows:

- The mass of each side concentrates at the vertex, namely, the mass of $X_{(i)}$ is $\frac{M}{N}$, where M is the total mass of the material.
- For each vertex $X_{(i)}$ only two tensions from $X_{(i-1)}$ and $X_{(i+1)}$ are acted. The tension $F_{(i+1)}$ from $X_{(i+1)}$ is represented by

$$F_{(i+1)} = f(\varepsilon_{(i)}) \frac{X_{(i+1)} - X_{(i)}}{l_{(i)}}.$$

Here, the stress function $f(\varepsilon_{(i)})$ and $\frac{X_{(i+1)} - X_{(i)}}{l_{(i)}}$ correspond to the magnitude and the direction of the tension, respectively.

- We assume that

$$f(\varepsilon) = \frac{\kappa}{2} \left(\varepsilon + \frac{1}{2} - \frac{1}{2(\varepsilon + 1)^2} \right) \quad \text{for the strain } \varepsilon > -1, \quad (1.3.1)$$

where $\kappa > 0$.

One of purposes in Chapter 2 is to construct a numerical scheme by applying the structure preserving numerical method (see Furihata and Matsuo [14]). By existence of the direction $\frac{X_{(i+1)} - X_{(i)}}{l_{(i)}}$, we could not show that the numerical scheme has solutions, when the stress function is linear, namely, Hooke's law is assumed. Hence, in order to overcome this difficulty we propose the nonlinear stress function f having a singularity at $\varepsilon = -1$. The singularity of f at $\varepsilon = -1$ means that the magnitude tends to infinity as the strain goes to -1 . Moreover, we choose coefficients appearing in (1.3.1) such that $f'(0) = \kappa$, namely, for small ε the behavior of f is similar to the function defined by Hooke's law. We will discuss about this kind of singularities at the later part of this chapter.

Under the physical assumption as above, Newton's law implies the following initial value problem for the system of second order ordinary differential equations. We denote this initial value problem by (OP)($X^{(0)}, V^{(0)}$): For all $T > 0$,

$$m \frac{d^2 X_{(i)}}{dt^2} = f(\varepsilon_{(i)}) \frac{X_{(i+1)} - X_{(i)}}{l_{(i)}} - f(\varepsilon_{(i-1)}) \frac{X_{(i)} - X_{(i-1)}}{l_{(i-1)}} \quad \text{on } [0, T], \quad (1.3.2)$$

$$\frac{dX_{(i)}}{dt} = V_{(i)}, \quad (1.3.3)$$

$$X_{(i)}(0) = X_{0(i)}, \quad \frac{d}{dt} X_{(i)}(0) = V_{0(i)} \quad \text{for } i = 1, 2, \dots, N, \quad (1.3.4)$$

where $X_{(N)} = X_{(0)}$ and $X_{(N+1)} = X_{(1)}$, $X^{(0)} = (X_{0(1)}, X_{0(2)}, \dots, X_{0(N)})$, $V^{(0)} = (V_{0(1)}, V_{0(2)}, \dots, V_{0(N)})$, $X_{0(i)}$ is the initial position and $V_{0(i)}$ is the initial velocity for $i = 1, 2, \dots, N$. We put $m = \frac{M}{N}$. In the above system f indicates the stress function and is given by (1.3.1).

One of purposes in this chapter is to discuss a numerical scheme to obtain approximate solutions of (OP)($X^{(0)}, V^{(0)}$) such that the following energy $F(X, V)$ is conserved:

$$F(X, V) = \sum_{i=1}^N \left\{ \frac{m}{2} |V_{(i)}|^2 + l_{N*} \hat{f}(\varepsilon_{(i)}) \right\} \quad \text{for } (X, V) \in \mathbb{R}^{4N},$$

where $m = \frac{M}{N}$, $\hat{f}(\varepsilon_{(i)}) = \frac{\kappa}{4} \left(\varepsilon_{(i)}^2 + \varepsilon_{(i)} + \frac{1}{1 + \varepsilon_{(i)}} \right)$ for $i = 1, 2, \dots, N$,

$X = (X_{(1)}, X_{(2)}, \dots, X_{(N)})$ and $V = (V_{(1)}, V_{(2)}, \dots, V_{(N)})$. By applying ideas given in [14], we get the following numerical scheme (NS) = (NS)($\Delta t, X^{(n)}, V^{(n)}$), where $\Delta t = \frac{T}{K}$ for $K \in \mathbb{Z}_{>0}$. Find $X^{(n+1)} = (X_{(1)}^{(n+1)}, X_{(2)}^{(n+1)}, \dots, X_{(N)}^{(n+1)})$ and $V^{(n+1)} = (V_{(1)}^{(n+1)}, V_{(2)}^{(n+1)}, \dots, V_{(N)}^{(n+1)})$

such that

$$\begin{aligned} & \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \\ &= -\frac{\kappa}{4m} \left\{ \left\{ \varepsilon_{(i-1)}^{(n+1)} + \varepsilon_{(i-1)}^{(n)} + 1 - \frac{1}{(1 + \varepsilon_{(i-1)}^{(n+1)}) (1 + \varepsilon_{(i-1)}^{(n)})} \right\} \frac{X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{|X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \right. \\ & \quad \left. - \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{(1 + \varepsilon_{(i)}^{(n+1)}) (1 + \varepsilon_{(i)}^{(n)})} \right\} \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{|X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right\}, \quad (1.3.5) \end{aligned}$$

$$\frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t} = \frac{V_{(i)}^{(n+1)} + V_{(i)}^{(n)}}{2} \quad \text{for } n = 0, 1, \dots, K-1 \text{ and } i = 1, 2, \dots, N, \quad (1.3.6)$$

where $X^{(n)}, V^{(n)}$ and $\varepsilon^{(n)} = (\varepsilon_{(1)}^{(n)}, \varepsilon_{(2)}^{(n)}, \dots, \varepsilon_{(N)}^{(n)})$ are given, and $X_{(0)}^{(n)} = X_{(N)}^{(n)}$ and $X_{(N+1)}^{(n)} = X_{(1)}^{(n)}$ for any $n = 0, 1, \dots, K-1$.

In this chapter, we establish existence and uniqueness of solutions to (OP)($X^{(0)}, V^{(0)}$) by applying the Banach fixed point theorem and some uniform estimates in Section 2.2. Also, we discuss existence of solutions to the scheme (NS) in Section 2.3. Moreover, we show convergence of the numerical solutions to the solution of (OP)($X^{(0)}, V^{(0)}$) and its convergence rate.

We note that our system has two advantages. In mathematical analysis for dynamics of elastic materials, it is important to estimate lower bounds for the strain ε , since $\varepsilon = -1$ means that the length vanishes. In this thesis, by applying the singularity of the stress function, we can get the lower bound of ε (see (2.2.7) in Lemma 2.2). This is the first advantage of this research. As mentioned later, we can get similar estimates on the low bounds in the system of partial differential equations, too. The second one is that we can construct the numerical scheme such that the energy is conserved, even though it is not easy to handle the difference of $\frac{X_{(i+1)} - X_{(i)}}{l_{(i)}}$. To the system for

shape memory alloys Yoshikawa [34, 36] already applied the structure preserving numerical method and discussed error estimates. Our numerical results to (OP)($X^{(0)}, V^{(0)}$) will be shown in Section 2.6.

Moreover, in Theorem 2.2 we provide a numerical scheme to obtain approximate solutions of (OP) by applying the discrete variational derivative method (DVDM) which enables us to derive the structure-preserving scheme systematically (see for instance [14]). The idea of applying DVDM to analyze elastic materials was found in [33, 34, 35, 36]. Here, we remark that the nonlinear implicit formula obtained by DVDM leads to expensive calculation costs, since some iterative schemes such as the Newton method are used. In order to overcome this difficulty, Matsuo and Furihata [23] have constructed linearly implicit structure-preserving schemes for complex-valued nonlinear PDEs by introducing extra time steps of numerical schemes, and Furihata has designed explicit structure-preserving schemes for nonlinear wave equations in [12]. Also, Matsuo [22] has constructed new implicit and explicit structure-preserving schemes for second-order PDEs by representing the second-order ones as a system of first-order ones and discretizing the system using DVDM. Furthermore, other results of linearly implicit structure-preserving ones have been obtained in [13] and [16]. Besides, a similar framework by Furihata, Matsuo, and collaborators has also been developed by Dahlby and Owren [10]. For these results, we note that a linearly implicit scheme obtained by DVDM can be unstable depending on the discretization of the energy. Recently, for this issue,

Matsuo and Furihata [24] and Sato et al. [29] have obtained the results of the analysis of the asymptotic behavior of the dissipative multi-steps linearly implicit schemes for the gradient system and the Duffing equation. By applying this idea, we propose a new multi-steps explicit scheme for (OP) in Section 2.7.

In Section 2.6 we compare the accuracies of the numerical scheme (NS) and the multi-steps explicit scheme (MS) for (OP). The exact formulation of (NS) and (MS) are given in (1.3.5), (1.3.6) and Section 2.7, respectively. In order to compare the accuracies of these methods we observe the periodic behaviors for the solutions of (OP). Moreover, in Section 2.5 we prove existence of a periodic solution of (OP) in time as Proposition 2.1, when initial states $X^{(0)}$ and $V^{(0)}$ are spherically symmetric. For proving it we introduce the initial value problem $P(R_0, v_0)$ for the one-dimensional ordinary differential equation:

$$m \frac{d^2 R}{dt^2} = -\beta g(R) \text{ on } [0, T], R(0) = R_0 \text{ and } \frac{dR}{dt}(0) = v_0, \quad (1.3.7)$$

where $g(R) = f(\alpha R - 1)$, α and β are constants, R_0 is a positive constant and $v_0 \in \mathbb{R}$. Here, we note that we show that the one-dimensional problem (1.3.7) has a solution and the solution is always periodic in time. Moreover, in numerical result to (1.3.7) the approximation seems to be periodic. In order to discuss the accuracies of (NS) and (MS), we compare the periodic stabilities of numerical solutions obtained by (NS) and (MS) in Sections 2.6 and 2.7, respectively.

In Chapters 3, 4, we consider initial and boundary value problems for beam equations with the nonlinear strain. At the beginning of Chapter 3, we derive a partial differential equation by letting $N \rightarrow \infty$ in (OP), and by adding the forth derivative term in space, we propose the partial differential equation called a beam equation in Section 3.1. Also, in order to clarify the role of the fourth derivative term we observe the numerical results in varying value of its coefficient in Section 3.5.

As mentioned above, the mathematical model representing stretching and shrinking motions of the curve made of elastic materials on the plane has been proposed. The model is given by:

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\partial u}{\partial x} \right) - \mu \frac{\partial^3 u}{\partial t \partial x^2} &= 0, \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \quad \text{on } Q(T), \\ \frac{\partial^i}{\partial x^i} u(0) &= \frac{\partial^i}{\partial x^i} u(1) \quad \text{on } (0, T) \text{ for any } i = 0, 1, 2, 3, \\ u(0) &= u_0, \quad \frac{\partial}{\partial t} u(0) = v_0 \quad \text{on } (0, 1), \end{aligned}$$

where ρ is the positive constant representing the density, γ is the positive constant and μ is the non-negative constant. Also, ε is the strain and f is the stress function. We note that we add the viscosity term describing energy decay to the equation. Throughout this thesis we denote this initial and boundary value problems by $P_0(u_0, v_0)$ if $\mu = 0$ and $P_\mu(u_0, v_0)$ if $\mu > 0$. We consider $P_0(u_0, v_0)$ in Chapter 3 and $P_\mu(u_0, v_0)$ in Chapter 4 respectively.

In Chapter 3, we consider the following initial and boundary problem $P_0(u_0, v_0)$ with having a Lipschitz continuous stress function:

$$\rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\partial u}{\partial x} \right) = 0, \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \quad \text{on } Q(T), \quad (1.3.8)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \quad \text{on } (0, T) \text{ for any } i = 0, 1, 2, 3, \quad (1.3.9)$$

$$u(0) = u_0, \quad \frac{\partial}{\partial t} u(0) = v_0 \quad \text{on } (0, 1), \quad (1.3.10)$$

where the stress function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous on \mathbb{R} . The aim of this chapter is to establish uniqueness and existence of weak solutions. In our formulation the strain is given by $\varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1$, namely, the strain is nonlinear with respect to $\frac{\partial u}{\partial x}$, and is not differentiable at $\frac{\partial u}{\partial x} = 0$. We note that this singularity arises from the dimensional difference between the domain and the range of the strain. Accordingly, the third term in the left hand side of (1.3.8) may not be a function. Hence, we consider only weak solutions for P_0 with the Lipschitz continuous stress function f .

The existence of weak solutions to P_0 is proved by the Galerkin method. Accordingly, first in the proof we construct a basis $\{\psi_n\}_{n=1}^\infty$ in $H = L^2(0, 1)^2$ and approximate H by the subspace generating by finite elements of the basis. Also, we approximate a solution of P_0 by $u^{(n)}(t, x) = \sum_{k=1}^n a_k^{(n)}(t) \psi_k(x)$,

and obtain a system consisting of ordinary differential equations for $a_k^{(n)}$. The Banach fixed point theorem and standard way to get uniform estimates implies the existence of approximate solutions $u^{(n)}$ and its convergence. Thus, we can prove the existence in Chapter 3. In particular, the uniqueness is proved by applying the approximate dual equation method, since for weak solution u the second derivative u_{tt} with respect to time is not a function and the u_t does not work well as a test function for proving the uniqueness. The dual equation method was investigated by Ladyženskaja, Solonnikov and Ural'ceva in [19] to prove uniqueness of weak solutions to parabolic and hyperbolic equations. Also, Niezgódka and Pawlow in [21] have improved the method by utilizing approximation of solutions to the dual problem in order to prove uniqueness for solutions to multi-dimensional Stefan problem. Moreover, this idea was applied to shape memory alloy problems in [2]. Thus, the approximation dual equation methods is known as a powerful way to prove uniqueness of solutions with low regularity.

In Chapter 3, we show existence of solutions of the dual equations, in detail, even if the dual equation is linear, since our equations are beam equations for which there is few literature dealing with strong solutions. In Section 3.3 we establish existence of strong solution to the linear problem and the result is applied in Chapter 4.

In Chapter 4, we consider the following initial and boundary problem $P_\mu(u_0, v_0)$ with the viscosity term and the compressible stress function for the nonlinear strain for $\mu > 0$:

$$\rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\partial u}{\partial x} \right) - \mu \frac{\partial^3 u}{\partial t \partial x^2} = 0, \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \quad \text{on } Q(T), \quad (1.3.11)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \quad \text{on } (0, T) \text{ for any } i = 0, 1, 2, 3, \quad (1.3.12)$$

$$u(0) = u_0, \quad \frac{\partial}{\partial t} u(0) = v_0 \quad \text{on } (0, 1), \quad (1.3.13)$$

where $f = f(\varepsilon)$ is the compressible stress function from $(-1, \infty)$ to \mathbb{R} having a singularity point at $\varepsilon = -1$. Precisely, we assume that this stress function $f : (-1, \infty) \rightarrow \mathbb{R}$ is given by

$$f(\varepsilon) = \frac{\kappa}{4} \left(1 - \frac{1}{(1 + \varepsilon)^4} \right) \quad \text{for } \varepsilon > -1, \quad (1.3.14)$$

where κ is the positive constant. Our aims of Chapter 4 are to prove theorems on existence and uniqueness for weak and strong solutions of P_μ accompanying the compressible stress function f given in (1.3.14). The key in the proofs is Lemma 4.3 which guarantees the lower bounds for the strain as follows: Let $z \in H^2(0, 1)^2$ and $K_1, K_2 > 0$. If

$$\int_0^1 \frac{1}{|z_x|^2} dx \leq K_1, |z_{xx}|_H \leq K_2,$$

then we obtain

$$|z_x| \geq \frac{K_2}{\sqrt{2}} e^{-K_1 K_2^2} \quad \text{on } [0, 1].$$

From Lemma 4.3 for solutions of P_μ we can take a positive constant δ such that

$$|u_x| \geq \delta \text{ on } \overline{Q(T)},$$

this shows that $f(\varepsilon)$ is sufficiently smooth on $\overline{Q(T)}$. Hence, we can prove existence of solutions to P_μ for $\mu > 0$. For the proof of existence, we first solve a linear equation by applying time discretization method and the Riesz representation theorem. We note that a proof of this part is omitted, since it is proved by a similar way to that given in Chapter 3. Next, by applying the Banach fixed point theorem, we can obtain a strong solution of P_μ , and from uniform estimates for strong solutions we prove existence of weak solutions.

On the uniqueness, thanks to the viscosity term, the inequality (Lemma 4.1) for the difference of weak solutions of the linear beam equation works well. Hence, we can prove the uniqueness by the standard method.

As mentioned above, we propose the problems (OP), P_0 and P_μ which are mathematical models for dynamics of elastic curers. Here, we summaries features of these models as follows:

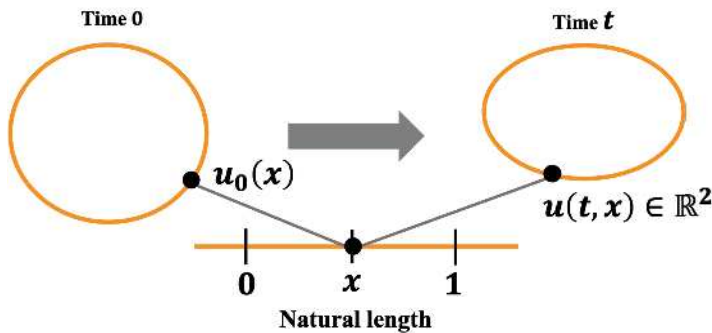


Figure 1.7: Unknown function u

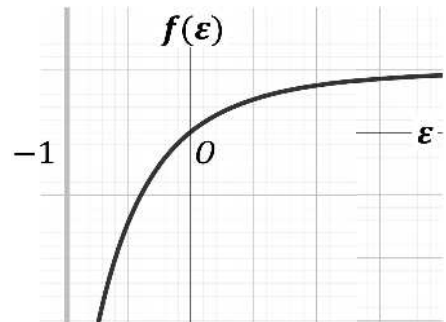


Figure 1.8: Compressible stress function f

- i) (Unknown function) We emphasize that the unknown function of the differential equation for elastic materials usually is the displacement, however, in these models u represents the position in \mathbb{R}^2 at time $t \in [0, T]$ and $x \in [0, 1]$ (see Figure 1.7). Our research aim is to represent motions of the elastic curve directly, therefore we adopt the position u as the unknown function, not the displacement. Also, we note that it is not possible to define a displacement in our system in the usual sense. Indeed, the displacement is given by the difference $U(t) - U_0$ between the original position U_0 and the position $U(t)$ at time t . However, in our present system the difference is not defined well, because of the dimensional difference. Based on the above argument we choose the position as the unknown function of our model.
- ii) (Nonlinear strain) In this thesis we define the strain ε by $\varepsilon = |u_x| - 1$ for the PDE model. This strain expresses the ratio between the length of stretching and its original length. Since we describe the motion of the one dimensional material on \mathbb{R}^2 , such nonlinear strain appears. Here, we note that $|u_x|$ may vanish in general and in this case it is impossible to calculate the derivative of ε with respect to x . Hence, in Chapter 3 we consider only weak solutions to avoid the differentiability of ε .
- iii) (Stress functions in P_0 and P_μ) In this thesis the three types of stress functions appear, namely, for (OP) $f_1(\varepsilon) = \frac{\kappa}{2} \left(\varepsilon + \frac{1}{2} - \frac{1}{2(1+\varepsilon)^2} \right)$ for $\varepsilon > -1$, for P_0 f is Lipschitz continuous on \mathbb{R} and for P_μ

$$f_2(\varepsilon) = \frac{\kappa}{4} \left(1 - \frac{1}{(1+\varepsilon)^4} \right) \text{ for } \varepsilon > -1.$$

we note that f_1 and $f_2 \rightarrow -\infty$ as $\varepsilon \rightarrow -1$. This divergence means that the stress tends to be extremely large, if we compress an elastic material to be one point.

The material having this property is called the compressible elastic body (see [15, 25, 30]). Here, by referring Section 6.5 in [15], we introduce the following examples of the stress function $f(\varepsilon)$.

$$f(\varepsilon) = \frac{\kappa}{\beta^2} \left(\frac{\beta}{\varepsilon} - \frac{1}{\beta} \varepsilon^{-\beta-1} \right), \frac{\kappa}{2} \left(\varepsilon - \frac{1}{\varepsilon} \right),$$

where κ, β are positive constants. In particular, $\beta = 9$ is shown as the specific value of β .

Based on these previous results we assume such a singularity for the stress function and decide f_1 and f_2 such that the mathematical treatment for solutions is easy. Particularly, the specific form of f_1 is chosen by considering that numerical computation would also be easier.

- iv) (Fourth derivative term) Our partial differential equation models contain fourth derivative term γu_{xxxx} where role is to approximate the kinetic equation. By effect of this term we can get enough regularity in space for the solutions and establish existence and uniqueness. On the other hand, due to [9], this term is regarded as a description of non-local effect induced by interfacial energy. In order to clarify the role of the term we compute numerical solutions in varying the value of the constant γ in Section 3.5.

In Chapter 5, we give future work related to our results.

On the ODE model, we point out some issues obtained from observation for our numerical simulation results, and in order to solve them we list some tasks concerned with development of new numerical schemes.

For the PDE models, main issues are to show large time behavior of solutions and to generalize

the existence and uniqueness theorems mentioned in Chapters 3 and 4.

It is a natural question to analyze the asymptotic behavior of the solution $u(t)$ as time t goes to infinity, from mathematical point of view. Indeed, for P_μ , $\mu > 0$, since we assume the energy decay by the viscosity term, we can expect convergence of the orbit $\{u(t)|t \geq 0\}$. Also, for this kind of analysis it is useful to clarify the properties of the stationary solutions to P_μ , $\mu > 0$. Hence, investigation to the stationary problem is one of the future tasks on this subject.

On the second issue for the PDE model, we note that our class of the stress functions does not cover the examples studied in the previous results based on the mathematical engineering, since throughout this thesis the stress function for the compressible elastic material are given by the specific forms. Subsequently, we aim to generalize the theorems concerned with existence and uniqueness of solutions, in order to deal with the examples.

Finally, as far future work, we mention about the rapid rotational motion of the elastic rings and the obstacle problem for elastic materials.

Chapter 2

The Ordinary differential equation model with stress function having singularity

In this chapter, we consider the initial value problem (OP)($X^{(0)}, V^{(0)}$) and the numerical scheme (NS) which are given in Chapter 1.

First, we show our mathematical results concerned with (OP) and (NS).

2.1 Mathematical results

The first mathematical result is concerned with existence and uniqueness of solutions to (OP)($X^{(0)}, V^{(0)}$).

Theorem 2.1. *Let f be a function given by (1.3.1), $X^{(0)} = (X_{0(1)}, X_{0(2)}, \dots, X_{0(N)})$, $V^{(0)} = (V_{0(1)}, V_{0(2)}, \dots, V_{0(N)})$, $X_{0(i)} \in \mathbb{R}^2$ and $V_{0(i)} \in \mathbb{R}^2$ for $i = 1, 2, \dots, N$. If $X_{0(i)} \neq X_{0(i+1)}$ for $i = 1, 2, \dots, N$, then (OP)($X^{(0)}, V^{(0)}$) has a unique solution $X = (X_{(1)}, X_{(2)}, \dots, X_{(N)}) \in C^2([0, T])^{2N}$.*

Theorem 2.2 represents existence of a unique solution for the numerical scheme (NS).

Theorem 2.2. *Let f be a function given by (1.3.1), $X_{0(i)} \in \mathbb{R}^2$ and $V_{0(i)} \in \mathbb{R}^2$ for $i = 1, 2, \dots, N$. If $X_{0(i)} \neq X_{0(i+1)}$ for $i = 1, 2, \dots, N$, then there exists $K_0 \in \mathbb{Z}_{>0}$ such that (NS)($\Delta t, X^{(n)}, V^{(n)}$) has a unique solution $(X^{(n+1)}, V^{(n+1)}) \in \mathbb{R}^4$ for $n = 0, 1, \dots, K - 1$ and $K \geq K_0$, where $\Delta t = \frac{T}{K}$ and $X^{(0)} = (X_{0(1)}, X_{0(2)}, \dots, X_{0(N)})$, $V^{(0)} = (V_{0(1)}, V_{0(2)}, \dots, V_{0(N)})$.*

The third result guarantees convergence of numerical solutions to the solution of (OP)($X^{(0)}, V^{(0)}$). f be a function given by (1.3.1).

Theorem 2.3. *Assume $X_{0(i)} \in \mathbb{R}^2$, $V_{0(i)} \in \mathbb{R}^2$ and $X_{0(i)} \neq X_{0(i+1)}$ for $i = 1, 2, \dots, N$. Let K_0 be a positive integer defined in Theorem 2.2 and $(X_K^{(n+1)}, V_K^{(n+1)})$ be a solution of (NS) $(\Delta t, X_K^{(n)}, V_K^{(n)})$*

for $n = 0, 1, \dots, K-1$ and $K \geq K_0$, where $\Delta t = \frac{T}{K}$, $X_K^{(0)} = (X_{0(1)}, X_{0(2)}, \dots, X_{0(N)})$ and $V_K^{(0)} = (V_{0(1)}, V_{0(2)}, \dots, V_{0(N)})$. Moreover, put

$$\begin{aligned} X^K(t) &= V_K^{(n)} \left(t - \frac{nT}{K} \right) + X_K^{(n)} \quad \text{for } \frac{nT}{K} < t \leq \frac{(n+1)T}{K} \text{ and } n = 0, 1, \dots, K-1, \\ X^K(0) &= X_0. \end{aligned}$$

There exists a positive constant C such that

$$\left| X(t) - X^K(t) \right| \leq C \left| \Delta t \right| \quad \text{for } 0 \leq t \leq T \text{ and } K \geq K_0,$$

where $X = (X_{(1)}, X_{(2)}, \dots, X_{(N)})$ is a solution of (OP)($X^{(0)}, V^{(0)}$).

In order to discuss the accuracy of the scheme (NS), we calculate approximations of periodic solutions in time. From now on, we consider (OP) in case the initial state is spherically symmetric. For $\theta = \frac{2\pi}{N}$ let A be the rotation matrix of angle θ and choose the initial values (see Figure 2.1) as follows:

$$X_{0(i)} = R_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V_{0(i)} = v_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } i = 1, 2, \dots, N. \quad (2.1.1)$$

Also, we denote by $P(R_0, v_0)$ the following initial value problem for the 1-dimensional ordinary differential equation, where $R_0 > 0$ and $v_0 \in \mathbb{R}$.

Problem $P(R_0, v_0)$: Find a function R on $[0, T]$ satisfying

$$m \frac{d^2 R(t)}{dt^2} = -\beta g(R(t)) \quad \text{for } t \in [0, T], \quad R(0) = R_0, \quad \frac{dR}{dt}(0) = v_0,$$

where R indicates the distance between the vertex at time t , $\alpha = \frac{2}{l_{N*}} \sin \frac{\theta}{2}$, $\beta = 2 \sin \frac{\theta}{2}$ and $g(R) = f(\alpha R - 1)$.

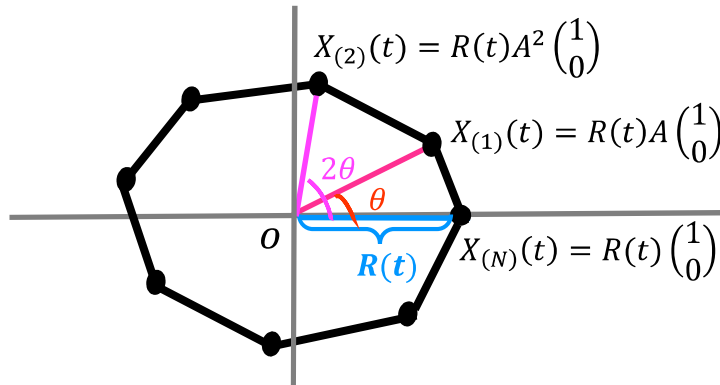


Figure 2.1: Spherically symmetric configuration

Here, let R be a solution of $P(R_0, v_0)$ on $[0, T]$ and put

$$X_i(t) = R(t)A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for all } i = 1, 2, \dots, N \text{ and } t \in [0, T].$$

Under the condition (2.1.1) by putting $\varepsilon = (\beta R - l_{N*})/l_{N*}$ on $[0, T]$ it is easy to see that $X = (X_{(1)}, X_{(2)}, \dots, X_{(N)})$ is a solution of (OP) on $[0, T]$. As seen in Figures 2.2 and 2.3 obtained by numerical simulations with $N = 12$, $m = 5/6$, the time step $\Delta t = 0.001$ and $R_0 = 0.6$, the solution of $P(R_0, 0)$ seems to be periodic in time.

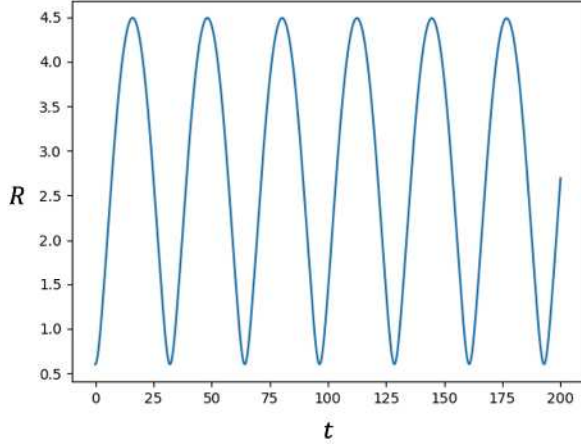


Figure 2.2: Time variation of R with $N = 12$ and $\Delta t = 0.001$

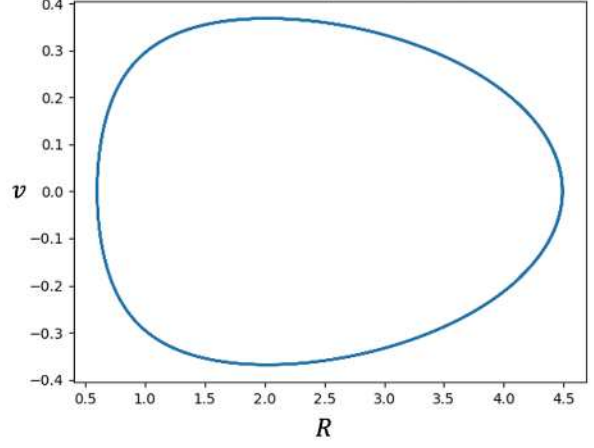


Figure 2.3: Phase diagram between R and $v := R'$ with $N = 12$ and $\Delta t = 0.001$

Moreover, we can obtain this property, mathematically, as follows.

Proposition 2.1. *If $R_0 > 0$, $v_0 \in \mathbb{R}$, then the solution of $P(R_0, v_0)$ is periodic in time.*

In Sections 2.2 – 2.5, we shall prove Theorems 2.1 – 2.3 and Proposition 2.1.

2.2 Existence and uniqueness

In this section, we prove Theorem 2.1. For its proof we need several steps. From (1.3.2) and (1.3.3) we can obtain the following integral equations

$$X_{(i)}(t_1) = \int_0^{t_1} V_{(i)}(t) dt + X_{0(i)}, \quad (2.2.1)$$

$$V_{(i)}(t_1) = \frac{1}{m} \int_0^{t_1} \left(f(\varepsilon_{(i)}) \frac{X_{(i+1)}(t) - X_{(i)}(t)}{l_{(i)}(t)} - f(\varepsilon_{(i-1)}) \frac{X_{(i)}(t) - X_{(i-1)}(t)}{l_{(i-1)}(t)} \right) dt + V_{0(i)}, \quad (2.2.2)$$

for $i = 1, 2, \dots, N$ and $t_1 \in [0, T]$.

Let $\mathcal{F}: C([0, T])^{4N} \rightarrow C([0, T])^{4N}$ be the mapping defined by the following

$$\mathcal{F}(Z) = (f_1(Z), f_2(Z), \dots, f_N(Z), g_1(Z), g_2(Z), \dots, g_N(Z)) \text{ for } Z = (X, V) \in C([0, T])^{4N}, \quad (2.2.3)$$

where

$$f_i(Z)(t) = \int_0^t V_{(i)}(\tau) d\tau + X_{0(i)}, \quad (2.2.4)$$

$$g_i(Z)(t) = \frac{1}{m} \left(\int_0^t f(\varepsilon_{(i)}) \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} - f(\varepsilon_{(i-1)}) \frac{X_{(i)}(\tau) - X_{(i-1)}(\tau)}{l_{(i-1)}(\tau)} d\tau \right) + V_{0(i)}, \quad (2.2.5)$$

$$X = (X_{(1)}, X_{(2)}, \dots, X_{(N)}), V = (V_{(1)}, V_{(2)}, \dots, V_{(N)}).$$

Moreover, we define the energy function F as follows

$$F(X, V) = \sum_{i=1}^N \left\{ \frac{m}{2} |V_{(i)}|^2 + l_{N*} \hat{f}(\varepsilon_{(i)}) \right\} \text{ for } (X, V) \in \mathbb{R}^{4N}, \quad (2.2.6)$$

$$\text{where } \hat{f}(\varepsilon_{(i)}) = \frac{\kappa}{4} \left(\varepsilon_i^2 + \varepsilon_{(i)} + \frac{1}{1 + \varepsilon_{(i)}} \right), \varepsilon_{(i)} = \frac{l_{(i)} - l_{N*}}{l_{N*}}, l_{(i)} = |X_{(i+1)} - X_{(i)}|$$

for $i = 1, 2, \dots, N$, $X = (X_{(1)}, X_{(2)}, \dots, X_{(N)})$ and $V = (V_{(1)}, V_{(2)}, \dots, V_{(N)})$.

The first lemma guarantees that the energy is preserved.

Lemma 2.1. *If X is a solution of $(OP)(X^{(0)}, V^{(0)})$, then F is preserved, namely,*

$$\frac{d}{dt} F(X(t), V(t)) = 0 \text{ for } t \in [0, T], \text{ where } V = \frac{dX}{dt} \text{ on } [0, T].$$

Proof. Let $t \in [0, T]$. By multiplying $\frac{d}{dt} X_{(i)}(t)$ on both sides of (1.3.2) for $i = 1, 2, \dots, N$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{m}{2} \sum_{i=1}^N \left| \frac{d}{dt} X_{(i)}(t) \right|^2 \right) \\ &= \sum_{i=1}^N f(\varepsilon_{(i)}(t)) \frac{X_{(i+1)}(t) - X_{(i)}(t)}{l_{(i)}(t)} \left(\frac{d}{dt} X_{(i)}(t) \right) - \sum_{i=1}^N f(\varepsilon_{(i-1)}(t)) \frac{X_{(i)}(t) - X_{(i-1)}(t)}{l_{(i-1)}(t)} \left(\frac{d}{dt} X_{(i)}(t) \right) \\ &= \sum_{i=1}^N f(\varepsilon_{(i)}(t)) \frac{X_{(i+1)}(t) - X_{(i)}(t)}{l_{(i)}(t)} \left(\frac{d}{dt} X_{(i)}(t) \right) - \sum_{i=1}^N f(\varepsilon_{(i)}(t)) \frac{X_{(i+1)}(t) - X_{(i)}(t)}{l_{(i)}(t)} \left(\frac{d}{dt} X_{(i+1)}(t) \right) \\ &= - \sum_{i=1}^N \frac{f(\varepsilon_{(i)}(t))}{2l_{(i)}(t)} \frac{d}{dt} |X_{(i+1)}(t) - X_{(i)}(t)|^2. \end{aligned}$$

In the calculation above, by using the condition $X_{(N)} = X_{(0)}$ and $X_{(N+1)} = X_{(1)}$, we obtain the second equation. Moreover, by substituting $l_{(i)}(t) = |X_{(i+1)}(t) - X_{(i)}(t)|$ and $\varepsilon_{(i)}(t) = \frac{l_{(i)}(t) - l_{N*}}{l_{N*}}$ for $i = 1, 2, \dots, N$ and $t \in [0, T]$, we have

$$\frac{f(\varepsilon_{(i)}(t))}{2l_{(i)}(t)} \frac{d}{dt} |X_{(i+1)}(t) - X_{(i)}(t)|^2 = l_{N*} f(\varepsilon_{(i)}) \frac{d}{dt} \varepsilon_{(i)}(t) \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T].$$

Since $\hat{f}(\varepsilon) = \frac{\kappa}{4} \left(\varepsilon^2 + \varepsilon + \frac{1}{1 + \varepsilon} \right)$ is the primitive of f , we have

$$\frac{d}{dt} \left\{ \sum_{i=1}^N \left(\frac{m}{2} \left| \frac{dX_{(i)}}{dt}(t) \right|^2 + l_{N*} \frac{d}{dt} \hat{f}(\varepsilon_{(i)}) \right) \right\} = 0 \quad \text{for } t \in [0, T],$$

and

$$\frac{d}{dt} \left\{ \sum_{i=1}^N \left(\frac{m}{2} |V_{(i)}(t)|^2 + l_{N*} \frac{d}{dt} \hat{f}(\varepsilon_{(i)}) \right) \right\} = 0 \quad \text{for } t \in [0, T],$$

namely,

$$\frac{d}{dt} F(X, V) = 0 \quad \text{on } [0, T].$$

Hence, F is preserved. Thus, Lemma 2.1 is proved. \square

Next, we give some uniform estimates for X and V obtained from the energy F .

Lemma 2.2. *Let $(X, V) \in \mathbb{R}^{4N}$. If $d_0 = F(X, V) \in \mathbb{R}$, then the following inequalities hold*

$$|X_{(i+1)} - X_{(i)}| \geq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}}, \quad (2.2.7)$$

$$|X_{(i+1)} - X_{(i)}| \leq \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}), \quad (2.2.8)$$

$$|V_{(i)}| \leq \sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \quad \text{for } i = 1, 2, \dots, N, \quad (2.2.9)$$

where $X = (X_{(1)}, X_{(2)}, \dots, X_{(N)})$, $V = (V_{(1)}, V_{(2)}, \dots, V_{(N)})$, $l_{(i)} = |X_{(i+1)} - X_{(i)}|$,

$\varepsilon_{(i)} = \frac{l_{(i)} - l_{N*}}{l_{N*}}$ and $X_{(N+1)} = X_{(1)}$.

Proof. First, let us show (2.2.7). By using (2.2.6) and $\varepsilon_{(i)} > -1$ for $i = 1, 2, \dots, N$, we have

$$\begin{aligned} d_0 &= F(X, V) \\ &\geq l_{N*} \sum_{i=1}^N \hat{f}(\varepsilon_{(i)}) \\ &\geq \frac{\kappa l_{N*}}{4} \sum_{i=1}^N \left(\varepsilon_{(i)} + \frac{1}{1 + \varepsilon_{(i)}} \right) \\ &\geq \frac{\kappa l_{N*}}{4} \left(-N + \frac{l_{N*}}{l_{(i)}} \right) \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

Therefore, we obtain

$$\frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} \leq l_{(i)} = |X_{(i+1)} - X_{(i)}| \quad \text{for } i = 1, 2, \dots, N.$$

It means that (2.2.7) holds.

Next, we show (2.2.8). Since $\frac{l_{(i)}}{l_{N*}} > 0$ for $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
d_0 &= F(X, V) \\
&\geq l_{N*} \sum_{i=1}^N \hat{f}(\varepsilon_{(i)}) \\
&\geq \frac{\kappa l_{N*}}{4} \sum_{i=1}^N \varepsilon_{(i)} \\
&= \frac{\kappa l_{N*}}{4} \left(\sum_{i=1}^N \frac{l_{(i)}}{l_{N*}} - \sum_{i=1}^N 1 \right) \\
&\geq \frac{\kappa l_{N*}}{4} \left(\frac{l_{(i)}}{l_{N*}} - N \right) \quad \text{for } i = 1, 2, \dots, N.
\end{aligned}$$

Therefore, we obtain

$$|X_{(i+1)} - X_{(i)}| = l_{(i)} \leq \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) \quad \text{for } i = 1, 2, \dots, N,$$

so that (2.2.8) holds.

Finally, we show (2.2.9). By the similar way to (2.2.7), we have

$$\begin{aligned}
d_0 &= F(X, V) \\
&\geq \frac{m}{2} |V_{(i)}|^2 + l_{N*} \sum_{i=1}^N \hat{f}(\varepsilon_{(i)}(t)) \\
&\geq \frac{m}{2} |V_{(i)}|^2 + \frac{\kappa l_{N*}}{4} \sum_{i=1}^N \varepsilon_{(i)}(t) \\
&\geq \frac{m}{2} |V_{(i)}|^2 + \frac{\kappa l_{N*}}{4} \sum_{i=1}^N (-1) \\
&\geq \frac{m}{2} |V_{(i)}|^2 - \frac{N\kappa l_{N*}}{4} \quad \text{for } i = 1, 2, \dots, N.
\end{aligned}$$

Therefore, we obtain (2.2.9) as follows

$$|V_{(i)}| \leq \sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \quad \text{for } i = 1, 2, \dots, N.$$

Thus, Lemma 2.2 holds. □

The third lemma is concerned with existence of a set on which \mathcal{F} is a contraction mapping.

Lemma 2.3. *For $T_1 > 0$, $\delta > 0$, $M > 0$ and $M' > 0$ we put*

$$\begin{aligned}
W(T_1, \delta, M', M) &= \left\{ (X, V) \in C([0, T_1])^{4N} \mid \delta \leq |X_{(i+1)}(t) - X_{(i)}(t)| \leq M', \right. \\
&\quad \left. |V_{(i+1)}(t) - V_{(i)}(t)| \leq M \text{ for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1] \right\}.
\end{aligned}$$

If $X^{(0)} = (X_{0(1)}, X_{0(2)}, \dots, X_{0(N)}) \in \mathbb{R}^{2N}$, $V^{(0)} = (V_{0(1)}, V_{0(2)}, \dots, V_{0(N)}) \in \mathbb{R}^{2N}$ and $X_{0(i)} \neq X_{0(i+1)}$ for $i = 1, 2, \dots, N$, then there exists $T_0 \in (0, T]$ such that $\mathcal{F} : W(T_0, \delta, M', M) \rightarrow W(T_0, \delta, M', M)$ is a contraction.

Proof. First, we prove that \mathcal{F} is the function from $W(T_1, \delta, M', M)$ to $W(T_1, \delta, M', M)$ for some $T_1 \in (0, T]$ and $\delta, M, M' > 0$. Let $T_1 \in (0, T]$ and $(X, V) \in W(T_1, \delta, M', M)$. Since $X_{0(i)} \neq X_{0(i+1)}$ for $i = 1, 2, \dots, N$, it follows that $d_0 = F(X^{(0)}, V^{(0)}) \in \mathbb{R}$. Clearly, Lemma 2.2 implies

$$\begin{aligned} |f_{i+1}((X, V))(t) - f_i((X, V))(t)| &\geq |X_{0(i+1)} - X_{0(i)}| - \int_0^t |V_{(i+1)}(\tau) - V_{(i)}(\tau)| d\tau \\ &\geq |X_{0(i+1)} - X_{0(i)}| - MT_1 \\ &\geq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} - MT_1 \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1]. \end{aligned}$$

Now, we choose $\delta > 0$ such that $\frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} \geq 2\delta$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \geq 2\delta - MT_1 \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1].$$

And we choose $T_1 > 0$ such that $MT_1 \leq \delta$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \geq \delta \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1].$$

In the same way, by using (2.2.8), we have

$$\begin{aligned} &|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \\ &\leq \int_0^t |V_{(i+1)}(\tau) - V_{(i)}(\tau)| d\tau + |X_{0(i+1)} - X_{0(i)}| \\ &\leq MT_1 + |X_{0(i+1)} - X_{0(i)}| \\ &\leq MT_1 + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1]. \end{aligned}$$

Now, we choose $M' > 0$ such that $\frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) \leq \frac{M'}{2}$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \leq MT_1 + \frac{M'}{2} \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1].$$

And we choose $T_1 > 0$ such that $MT_1 \leq \frac{M'}{2}$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \leq M' \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1].$$

Hence, we choose $T_1 > 0$ such that $T_1 = \min \left\{ \frac{\delta}{M}, \frac{M'}{2M} \right\}$, we can obtain

$$\delta \leq |f_{i+1}((X, V))(t) - f_i((X, V))(t)| \leq M' \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1]. \quad (2.2.10)$$

Next, by using (2.2.5) and (2.2.9), we have

$$\begin{aligned}
& |g_{i+1}((X, V))(t) - g_i((X, V))(t)| \\
& \leq \frac{1}{m} \left| \int_0^t f(\varepsilon_{(i+1)}) \frac{X_{(i+2)}(\tau) - X_{(i+1)}(\tau)}{l_{(i+1)}(\tau)} d\tau \right| + \frac{2}{m} \left| \int_0^t f(\varepsilon_{(i)}) \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} d\tau \right| \\
& \quad + \frac{1}{m} \left| \int_0^t f(\varepsilon_{(i-1)}) \frac{X_{(i)}(\tau) - X_{(i-1)}(\tau)}{l_{(i-1)}(\tau)} d\tau \right| + |V_{0(i+1)}| + |V_{0(i)}| \\
& \leq \frac{1}{m} \left| \int_0^t f(\varepsilon_{(i+1)}) \frac{X_{(i+2)}(\tau) - X_{(i+1)}(\tau)}{l_{(i+1)}(\tau)} d\tau \right| + \frac{2}{m} \left| \int_0^t f(\varepsilon_{(i)}) \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} d\tau \right| \\
& \quad + \frac{1}{m} \left| \int_0^t f(\varepsilon_{(i-1)}) \frac{X_{(i)}(\tau) - X_{(i-1)}(\tau)}{l_{(i-1)}(\tau)} d\tau \right| + 2\sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \\
& =: A_1 + A_2 + A_3 + 2\sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T].
\end{aligned}$$

First, we choose $M > 0$ such that $2\sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \leq \frac{M}{5}$. Uniform estimates for A_i for $i = 1, 2, 3$ can be obtained in the same way, so we would like to show you how to get the uniform estimate for only A_1 . By using (1.3.1) and the definition of the strain, we have

$$\begin{aligned}
|A_1| &= \frac{\kappa}{4m} \left| \int_0^t \left\{ 2 \left(\frac{l_{(i+1)}(\tau) - l_{N*}}{l_{N*}} \right) + 1 + \left(\frac{l_{N*}}{l_{(i+1)}(\tau)} \right)^2 \right\} \frac{X_{(i+2)}(\tau) - X_{(i+1)}(\tau)}{|X_{(i+2)}(\tau) - X_{(i+1)}(\tau)|} d\tau \right| \\
&\leq \frac{\kappa}{4m} \int_0^t \left\{ 2 \left(\frac{|l_{(i+1)}(\tau)| + l_{N*}}{l_{N*}} \right) + 1 + \left| \frac{l_{N*}}{l_{(i+1)}(\tau)} \right|^2 \right\} d\tau.
\end{aligned}$$

Since $(X, V) \in W(T_1, \delta, M', M)$, we obtain

$$|A_1| \leq \frac{\kappa}{4m} \left\{ \frac{2}{l_{N*}} (M' + l_{N*}) + 1 + \frac{l_{N*}^2}{\delta^2} \right\} T_1.$$

Hence, we choose $T_1 > 0$ such that $\frac{\kappa}{4m} \left\{ \frac{2}{l_{N*}} (M' + l_{N*}) + 1 + \frac{l_{N*}^2}{\delta^2} \right\} T_1 \leq \frac{M}{5}$, we have

$$|A_1| \leq \frac{M}{5}.$$

In the same way, by choosing small $T_1 > 0$, we can obtain

$$|A_2| \leq \frac{2M}{5}, \quad |A_3| \leq \frac{M}{5}.$$

Hence, we have

$$|g_{i+1}((X, V))(t) - g_i((X, V))(t)| \leq \frac{M}{5} + \frac{2M}{5} + \frac{M}{5} + \frac{M}{5} = M \quad \text{for } i = 1, 2, \dots, N \text{ and } t \in [0, T_1].$$

By using the above uniform estimate and (2.2.10), we can prove $\mathcal{F}((X, V)) \in W(T_1, \delta, M', M)$, namely, \mathcal{F} is the mapping from $W(T_1, \delta, M', M)$ to $W(T_1, \delta, M', M)$.

Finally, we prove that $\mathcal{F} : W(T_0, \delta, M', M) \rightarrow W(T_0, \delta, M', M)$ is a contraction mapping for some $T_0 \in (0, T_1]$. Let $(X, V), (X', V') \in W(T, \delta, M', M)$. Since $X_{(i)}, V_{(i)}, f_i((X, V))$ and $g_i((X, V)) \in \mathbb{R}^2$ for $i = 1, 2, \dots, N$, we can put

$$\begin{aligned} X_{(i)} &= (X_{(i)1}, X_{(i)2}), & V_{(i)} &= (V_{(i)1}, V_{(i)2}), & f_i((X, V)) &= (f_{(i)1}((X, V)), f_{(i)2}((X, V))) \\ g_i((X, V)) &= (g_{(i)1}((X, V)), g_{(i)2}((X, V))) & & & & \text{for } i = 1, 2, \dots, N. \end{aligned}$$

And we regard the norm of $C([0, T])^{4N}$ as follows

$$\begin{aligned} & \|\mathcal{F}((X, V)) - \mathcal{F}((X', V'))\|_{C([0, T])^{4N}} \\ &= \left(\sum_{i=1}^N \|f_{(i)1}((X, V)) - f_{(i)1}((X', V'))\|_{C([0, T])}^2 + \sum_{i=1}^N \|f_{(i)2}((X, V)) - f_{(i)2}((X', V'))\|_{C([0, T])}^2 \right. \\ & \quad \left. + \sum_{i=1}^N \|g_{(i)1}((X, V)) - g_{(i)1}((X', V'))\|_{C([0, T])}^2 + \sum_{i=1}^N \|g_{(i)2}((X, V)) - g_{(i)2}((X', V'))\|_{C([0, T])}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.2.11)$$

By using (2.2.4), we have

$$\begin{aligned} & |f_{(i)j}((X, V))(t) - f_{(i)j}((X', V'))(t)| \\ & \leq \int_0^t |V_{(i)j}(\tau) - V'_{(i)j}(\tau)| d\tau \\ & \leq T_1 \|V_{(i)j} - V'_{(i)j}\|_{C([0, T_1])} \\ & \leq T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}} \quad \text{for } i = 1, 2, \dots, N, j = 1, 2 \text{ and } t \in [0, T_1]. \end{aligned}$$

Therefore, we can obtain

$$|f_{(i)j}((X, V))(t) - f_{(i)j}((X', V'))(t)| \leq T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}, \quad (2.2.12)$$

for $i = 1, 2, \dots, N$ and $t_1 \in [0, T]$.

Next, by using (2.2.5), we have

$$\begin{aligned} & |g_{(i)j}((X, V))(t) - g_{(i)j}((X', V'))(t)| \\ & \leq \frac{1}{m} \int_0^t \left| f(\varepsilon_{(i)}) \frac{X_{(i+1)j}(\tau) - X_{(i)j}(\tau)}{l_{(i)}(\tau)} - f(\varepsilon'_{(i)}) \frac{X_{(i+1)j}(\tau) - X_{(i)j}(\tau)}{l_{(i)}(\tau)} \right| d\tau \\ & \quad + \frac{1}{m} \int_0^t \left| f(\varepsilon'_{(i)}) \frac{X_{(i+1)j}(\tau) - X_{(i)j}(\tau)}{l_{(i)}(\tau)} - f(\varepsilon'_{(i)}) \frac{X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau)}{l_{(i)}(\tau)} \right| d\tau \\ & \quad + \frac{1}{m} \int_0^t \left| f(\varepsilon'_{(i)}) \frac{X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau)}{l_{(i)}(\tau)} - f(\varepsilon'_{(i)}) \frac{X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau)}{l'_{(i)}(\tau)} \right| d\tau \\ & \quad + \frac{1}{m} \int_0^t \left| f(\varepsilon_{(i-1)}) \frac{X_{(i)j}(\tau) - X_{(i-1)j}(\tau)}{l_{(i-1)}(\tau)} - f(\varepsilon'_{(i-1)}) \frac{X_{(i)j}(\tau) - X_{(i-1)j}(\tau)}{l_{(i-1)}(\tau)} \right| d\tau \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{m} \int_0^t \left| f(\varepsilon'_{(i-1)}) \frac{X_{(i)j}(\tau) - X_{(i-1)j}(\tau)}{l_{(i-1)}(\tau)} - f(\varepsilon'_{(i-1)}) \frac{X'_{(i)j}(\tau) - X'_{(i-1)j}(\tau)}{l_{(i-1)}(\tau)} \right| d\tau \\
& + \frac{1}{m} \int_0^t \left| f(\varepsilon'_{(i-1)}) \frac{X'_{(i)j}(\tau) - X'_{(i-1)j}(\tau)}{l_{(i-1)}(\tau)} - f(\varepsilon'_{(i-1)}) \frac{X'_{(i)j}(\tau) - X'_{(i-1)j}(\tau)}{l'_{(i-1)}(\tau)} \right| d\tau \\
& =: B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \quad \text{for } i = 1, 2, \dots, N, \quad j = 1, 2 \text{ and } t \in [0, T_1].
\end{aligned}$$

First, by using (1.3.1) and the definition of $\varepsilon_{(i)}$ and $l_{(i)}$ for $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
B_1 &= \int_0^t \left| \left\{ f(\varepsilon_{(i)}) - f(\varepsilon'_{(i)}) \right\} \frac{X_{(i+1)j}(\tau) - X_{(i)j}(\tau)}{l_{(i)}(\tau)} \right| d\tau \\
&\leq \frac{\kappa}{2m} \int_0^t \left| \varepsilon_{(i)}(\tau) - \varepsilon'_{(i)}(\tau) \right| \left| 1 + \frac{2 + \varepsilon_{(i)}(\tau) + \varepsilon'_{(i)}(\tau)}{2(1 + \varepsilon_{(i)}(\tau))^2(1 + \varepsilon'_{(i)}(\tau))^2} \right| \frac{|X_{(i+1)j}(\tau) - X_{(i)j}(\tau)|}{|l_{(i)}(\tau)|} d\tau \\
&\leq \frac{\kappa}{2m} \int_0^t \left| \frac{l_{(i)}(\tau) - l'_{(i)}(\tau)}{l_{N*}} \right| \left| 1 + \frac{l_{N*}^3(l_{(i)}(\tau) + l'_{(i)}(\tau))}{2l_{(i)}^2(\tau)l'_{(i)}(\tau)} \right| \frac{|X_{(i+1)j}(\tau) - X_{(i)j}(\tau)|}{|l_{(i)}(\tau)|} d\tau \\
&\leq \frac{\kappa}{2ml_{N*}} \int_0^t \left(|X_{(i+1)j}(\tau) - X'_{(i+1)j}(\tau)| + |X_{(i)j}(\tau) - X'_{(i)j}(\tau)| \right) \\
&\quad \times \left(1 + \frac{l_{N*}^3(|l_{(i)}(\tau)| + |l'_{(i)}(\tau)|)}{2|l_{(i)}(\tau)|^2|l'_{(i)}(\tau)|^2} \right) \frac{|X_{(i+1)j}(\tau) - X_{(i)j}(\tau)|}{|l_{(i)}(\tau)|} d\tau.
\end{aligned}$$

Since $(X, V) \in W(T_1, \delta, M', M)$, we have

$$\begin{aligned}
B_1 &\leq \frac{\kappa}{2ml_{N*}} \left(1 + \frac{l_{N*}^3 M'}{\delta^4} \right) \frac{M'}{\delta} \int_0^t \left(|X_{(i+1)j}(\tau) - X'_{(i+1)j}(\tau)| + |X_{(i)j}(\tau) - X'_{(i)j}(\tau)| \right) d\tau \\
&\leq \frac{\kappa M'}{m\delta l_{N*}} \left(1 + \frac{l_{N*}^3 M'}{\delta^4} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \tag{2.2.13}
\end{aligned}$$

In the same way, we can obtain

$$B_4 \leq \frac{\kappa M'}{m\delta l_{N*}} \left(1 + \frac{l_{N*}^3 M'}{\delta^4} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \tag{2.2.14}$$

Similarly, we see that

$$\begin{aligned}
B_2 &= \frac{1}{m} \int_0^t \left| \frac{f(\varepsilon'_{(i)})}{l_{(i)}(\tau)} \left(X_{(i+1)j}(\tau) - X_{(i)j}(\tau) - X'_{(i+1)j}(\tau) + X'_{(i)j}(\tau) \right) \right| d\tau \\
&\leq \frac{\kappa}{2m} \int_0^t \left(\frac{|l'_{(i)}(\tau)| + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{l_{N*}^2}{|l'_{(i)}(\tau)|^2} \right) \frac{1}{|l_{(i)}(\tau)|} \\
&\quad \times \left(|X_{(i+1)j}(\tau) - X'_{(i+1)j}(\tau)| + |X_{(i)j}(\tau) - X'_{(i)j}(\tau)| \right) d\tau \\
&\leq \frac{\kappa}{2m\delta} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{l_{N*}^2}{2\delta^2} \right) \int_0^t \left(|X_{(i+1)j}(\tau) - X'_{(i+1)j}(\tau)| + |X_{(i)j}(\tau) - X'_{(i)j}(\tau)| \right) d\tau
\end{aligned}$$

$$\leq \frac{\kappa}{m\delta} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{l_{N*}^2}{2\delta^2} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \quad (2.2.15)$$

Also, we can obtain

$$B_5 \leq \frac{\kappa}{m\delta} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{l_{N*}^2}{2\delta^2} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \quad (2.2.16)$$

Similarly to the argument above, we have

$$\begin{aligned} B_3 &= \frac{1}{m} \int_0^t \left| f(\varepsilon'_{(i)}) \left(X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau) \right) \left(\frac{1}{l_{(i)}(\tau)} - \frac{1}{l'_{(i)}(\tau)} \right) \right| d\tau \\ &\leq \frac{\kappa}{2m} \int_0^t \left(|\varepsilon'_{(i)}| + \frac{1}{2} + \frac{1}{2} \left| \frac{1}{(1 + \varepsilon'_{(i)})} \right|^2 \right) |X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau)| \frac{|l'_{(i)}(\tau) - l_{(i)}(\tau)|}{|l_{(i)}(\tau)| |l'_{(i)}(\tau)|} d\tau \\ &\leq \frac{\kappa}{2m} \int_0^t \left(\frac{|l'_{(i)}(\tau)| + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{|l'_{(i)}(\tau)|^2} \right) |X'_{(i+1)j}(\tau) - X'_{(i)j}(\tau)| \frac{|l'_{(i)}(\tau) - l_{(i)}(\tau)|}{|l_{(i)}(\tau)| |l'_{(i)}(\tau)|} d\tau \\ &\leq \frac{\kappa M'}{2m\delta^2} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{\delta^2} \right) \int_0^t |l'_{(i)}(\tau) - l_{(i)}(\tau)| d\tau \\ &\leq \frac{\kappa M'}{2m\delta^2} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{\delta^2} \right) \int_0^t \left(|X'_{(i+1)j}(\tau) - X_{(i+1)j}(\tau)| + |X'_{(i)j}(\tau) - X_{(i)j}(\tau)| \right) d\tau \\ &\leq \frac{\kappa M'}{m\delta^2} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{\delta^2} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \end{aligned} \quad (2.2.17)$$

In the same way, we can obtain

$$B_6 \leq \frac{\kappa M'}{m\delta^2} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{\delta^2} \right) T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}. \quad (2.2.18)$$

By using (2.2.13) - (2.2.18), we have

$$|g_{(i)j}((X, V))(t) - g_{(i)j}((X', V'))(t)| \leq \alpha T_1 \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}, \quad (2.2.19)$$

for $i = 1, 2, \dots, N$, $j = 1, 2$ and $t \in [0, T_1]$, where α is the positive constant given by

$$\alpha = \left\{ \frac{2\kappa M'}{m\delta l_{N*}} \left(1 + \frac{l_{N*}^3 M'}{\delta^4} \right) + \frac{2\kappa}{m\delta} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{l_{N*}^2}{2\delta^2} \right) + \frac{2\kappa M'}{m\delta^2} \left(\frac{M' + l_{N*}}{l_{N*}} + \frac{1}{2} + \frac{1}{2} \frac{l_{N*}^2}{\delta^2} \right) \right\}.$$

From (2.2.11), (2.2.12) and (2.2.19), it follows that

$$\begin{aligned} &\|\mathcal{F}((X, V)) - \mathcal{F}((X', V'))\|_{C([0, T_1])^{4N}} \\ &= \left(\sum_{i=1}^N \|f_{(i)1}((X, V)) - f_{(i)1}((X', V'))\|_{C([0, T_1])}^2 + \sum_{i=1}^N \|f_{(i)2}((X, V)) - f_{(i)2}((X', V'))\|_{C([0, T_1])}^2 \right. \\ &\quad \left. + \sum_{i=1}^N \|g_{(i)1}((X, V)) - g_{(i)1}((X', V'))\|_{C([0, T_1])}^2 + \sum_{i=1}^N \|g_{(i)2}((X, V)) - g_{(i)2}((X', V'))\|_{C([0, T_1])}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq T_1 \sqrt{2N(1+\alpha^2)} \|(X, V) - (X', V')\|_{C([0, T_1])^{4N}}.$$

Therefore, by choosing $T_0 \in (0, T_1]$ such that $T_0 < \frac{1}{\sqrt{2N(1+\alpha^2)}}$, we see that $\mathcal{F} : W(T_0, \delta, M', M) \rightarrow W(T_0, \delta, M', M)$ is a contraction. Hence, Lemma 2.3 is proved. \square

We can easily show the following lemma. So, we omit its proof.

Lemma 2.4. (1.3.2), (1.3.3), (1.3.4) and $X \in C^2([0, T])^{2N}$ hold if and only if (2.2.1), (2.2.2) and $(X, V) \in C^1([0, T])^{4N}$ hold.

Proof of Theorem 2.1. By Lemma 2.3, there exists $T_1 > 0$ such that $\mathcal{F} : W(T_1, \delta, M', M) \rightarrow W(T_1, \delta, M', M)$ is a contraction. The Banach fixed point theorem implies that there exists one and only one $Z \in W(T_1, \delta, M', M)$ such that $Z = \mathcal{F}(Z)$. Thus, by Lemma 2.4 (OP)($X^{(0)}, V^{(0)}$) has a unique solution on $[0, T_1]$.

Let $t \in [T_1, T_1 + T_2]$ where $T_2 \in (0, T - T_1]$. Let us consider the following initial value problem (OP) $^{(2)}(X_{(i)}^{(1)}(T_1), \frac{d}{dt}X_{(i)}^{(1)}(T_1))$:

$$\begin{aligned} m \frac{d^2 X_{(i)}^{(2)}}{dt^2}(t) &= f\left(\varepsilon_{(i)}^{(2)}\right) \frac{X_{(i+1)}^{(2)}(t) - X_{(i)}^{(2)}(t)}{l_{(i)}^{(2)}(t)} - f\left(\varepsilon_{(i-1)}^{(2)}\right) \frac{X_{(i)}^{(2)}(t) - X_{(i-1)}^{(2)}(t)}{l_{(i-1)}^{(2)}(t)} \text{ on } [T_1, T_1 + T_2], \\ X_{(i)}^{(2)}(T_1) &= X_{(i)}^{(1)}(T_1), \quad \frac{d}{dt}X_{(i)}^{(2)}(T_1) = \frac{d}{dt}X_{(i)}^{(1)}(T_1) \quad \text{for } i = 1, 2, \dots, N, \end{aligned}$$

where $X^{(1)}$ is the unique solution of (OP)($X^{(0)}, V^{(0)}$) on $[0, T_1]$. Here, we define $\mathcal{F} : C([T_1, T_1 + T_2])^{4N} \rightarrow C([T_1, T_1 + T_2])^{4N}$ in a similarly to (2.2.3)–(2.2.5). For $\delta > 0, M > 0$ and $M' > 0$ we put

$$\begin{aligned} W_1(T_2, \delta, M', M) &= \left\{ (X, V) \in C([T_1, T_1 + T_2])^{4N} \mid \delta \leq |X_{(i+1)}(t) - X_{(i)}(t)| \leq M', \right. \\ &\quad \left. |V_{(i+1)}(t) - V_{(i)}(t)| \leq M \text{ for } i = 1, 2, \dots, N, t \in [T_1, T_1 + T_2] \right\}. \end{aligned}$$

Here, since Lemma 2.1 implies $F(X^{(1)}(T_1), V^{(1)}(T_1)) = d_0$, we can show that

$\mathcal{F} : W_1(T_2, \delta, M', M) \rightarrow W_1(T_2, \delta, M', M)$ is a contraction. Hence, by applying the Banach fixed point theorem, again, we know that (OP)($X^{(0)}, V^{(0)}$) has a unique solution on $[0, 2T_1]$.

By repeating the discussion above, we obtain a solution of (OP)($X^{(0)}, V^{(0)}$) on $[0, T]$. Consequently, we can prove Theorem 2.1. \square

2.3 Numerical schemes

In this section, we prove Theorem 2.2. For its proof we need several steps. In order to handle (1.3.5) and (1.3.6) easily, for $K \in \mathbb{Z}_{>0}$ and $\Delta t = \frac{T}{K}$, we define $\hat{g}_{K(i)}(X^{(n+1)}, X^{(n)}, V^{(n)})$ and $\hat{f}_{K(i)}(V^{(n+1)}, V^{(n)}, X^{(n)})$ as follows

$$V_{(i)}^{(n+1)} = -\frac{\kappa \Delta t}{4m} \left\{ \varepsilon_{(i-1)}^{(n+1)} + \varepsilon_{(i-1)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i-1)}^{(n+1)}\right) \left(1 + \varepsilon_{(i-1)}^{(n)}\right)} \right\}$$

$$\begin{aligned}
& \times \frac{X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left|X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)}\right| + \left|X_{(i)}^{(n)} - X_{(i-1)}^{(n)}\right|} \\
& + \frac{\kappa \Delta t}{4m} \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(n+1)}\right) \left(1 + \varepsilon_{(i)}^{(n)}\right)} \right\} \\
& \times \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left|X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)}\right| + \left|X_{(i+1)}^{(n)} - X_{(i)}^{(n)}\right|} + V_{(i)}^{(n)} \\
& =: \hat{g}_{K(i)} \left(X^{(n+1)}, X^{(n)}, V^{(n)} \right), \tag{2.3.1}
\end{aligned}$$

$$\begin{aligned}
X_{(i)}^{(n+1)} &= \frac{\Delta t}{2} \left(V_{(i)}^{(n+1)} + V_{(i)}^{(n)} \right) + X_{(i)}^{(n)} \\
&=: \hat{f}_{K(i)} \left(V^{(n+1)}, V^{(n)}, X^{(n)} \right), \tag{2.3.2}
\end{aligned}$$

for $i = 1, 2, \dots, N, n = 0, 1, \dots, K$.

$X(0) = X^{(0)}, V(0) = V^{(0)}$. For given $X^{(n)}$ and $V^{(n)}$, let $\hat{\mathcal{F}}_K : \mathbb{R}^{4N} \rightarrow \mathbb{R}^{4N}$ be the following mapping

$$\begin{aligned}
\hat{\mathcal{F}}_K(Z) &= \left(\hat{f}_{K(1)} \left(V, V^{(n)}, X^{(n)} \right), \hat{f}_{K(2)} \left(V, V^{(n)}, X^{(n)} \right), \dots, \hat{f}_{K(N)} \left(V, V^{(n)}, X^{(n)} \right), \right. \\
&\quad \left. \hat{g}_{K(1)} \left(X, X^{(n)}, V^{(n)} \right), \hat{g}_{K(2)} \left(X, X^{(n)}, V^{(n)} \right), \dots, \hat{g}_{K(N)} \left(X, X^{(n)}, V^{(n)} \right) \right), \\
&\quad \text{for } Z = (X, V) \in \mathbb{R}^{4N}.
\end{aligned}$$

The first lemma guarantees that the energy is preserved by (NS).

Lemma 2.5. *If (NS) $\left(\Delta t, X^{(n)}, V^{(n)} \right)$ has a solution $\left(X^{(n+1)}, V^{(n+1)} \right)$ for $n = 0, 1, \dots, K-1$ and $K \in \mathbb{Z}_{>0}$, where $\Delta t = \frac{T}{K}$, then $F \left(X^{(n+1)}, V^{(n+1)} \right) = F \left(X^{(n)}, V^{(n)} \right)$ for $n = 0, 1, \dots, K-1$.*

Proof. For $n = 0, 1, \dots, K-1$, easily, we get

$$\begin{aligned}
& F \left(X^{(n+1)}, V^{(n+1)} \right) - F \left(X^{(n)}, V^{(n)} \right) \\
&= \frac{m}{2} \sum_{i=1}^N \left(\left| V_{(i)}^{(n+1)} \right|^2 - \left| V_{(i)}^{(n)} \right|^2 \right) + l_{N*} \sum_{i=1}^N \left(\hat{f} \left(\varepsilon_{(i)}^{(n+1)} \right) - \hat{f} \left(\varepsilon_{(i)}^{(n)} \right) \right) \\
&= \frac{m}{2} \sum_{i=1}^N \left(V_{(i)}^{(n+1)} - V_{(i)}^{(n)} \right) \left(V_{(i)}^{(n+1)} + V_{(i)}^{(n)} \right) + l_{N*} \sum_{i=1}^N \left(\hat{f} \left(\varepsilon_{(i)}^{(n+1)} \right) - \hat{f} \left(\varepsilon_{(i)}^{(n)} \right) \right).
\end{aligned}$$

By using (1.3.6), we have

$$\frac{m}{2} \sum_{i=1}^N \left(V_{(i)}^{(n+1)} - V_{(i)}^{(n)} \right) \left(V_{(i)}^{(n+1)} + V_{(i)}^{(n)} \right) = m \sum_{i=1}^N \left(V_{(i)}^{(n+1)} - V_{(i)}^{(n)} \right) \frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t}. \tag{2.3.3}$$

Here, by using the definition of \hat{f} , we have

$$\begin{aligned}
& \hat{f}\left(\varepsilon_{(i)}^{(n+1)}\right) - \hat{f}\left(\varepsilon_{(i)}^{(n)}\right) \\
&= \frac{\kappa}{4} \left\{ \left(\left(\varepsilon_{(i)}^{(n+1)} \right)^2 - \left(\varepsilon_{(i)}^{(n)} \right)^2 \right) + \left(\varepsilon_{(i)}^{(n+1)} - \varepsilon_{(i)}^{(n)} \right) + \left(\frac{1}{1 + \varepsilon_{(i)}^{(n+1)}} - \frac{1}{1 + \varepsilon_{(i)}^{(n)}} \right) \right\} \\
&=: \frac{\kappa}{4} (C_1 + C_2 + C_3).
\end{aligned}$$

Now, by using the definition of ε_i and l_i for $i = 1, 2, \dots, N$, we see that

$$\begin{aligned}
C_1 &= \left(\varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} \right) \left(\varepsilon_{(i)}^{(n+1)} - \varepsilon_{(i)}^{(n)} \right) \\
&= \frac{1}{l_{N*}} \left(\varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} \right) \left(l_{(i)}^{(n+1)} - l_{(i)}^{(n)} \right) \\
&= \frac{1}{l_{N*}} \left(\varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} \right) \frac{\left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right) \left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} - X_{(i+1)}^{(n)} + X_{(i)}^{(n)} \right)}{\left| X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|}.
\end{aligned}$$

Similarly, we observe that

$$\begin{aligned}
C_2 &= \frac{1}{l_{N*}} \left(l_{(i)}^{(n+1)} - l_{(i)}^{(n)} \right) \\
&= \frac{1}{l_{N*}} \frac{\left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right) \left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} - X_{(i+1)}^{(n)} + X_{(i)}^{(n)} \right)}{\left| X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|}, \\
C_3 &= -\frac{1}{\left(1 + \varepsilon_{(i)}^{(n+1)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right)} \left(\varepsilon_{(i)}^{(n+1)} - \varepsilon_{(i)}^{(n)} \right) \\
&= -\frac{1}{l_{N*}} \frac{\left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right) \left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} - X_{(i+1)}^{(n)} + X_{(i)}^{(n)} \right)}{\left(1 + \varepsilon_{(i)}^{(n+1)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right) \left| X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& l_{N*} \sum_{i=1}^N \left\{ \hat{f}\left(\varepsilon_{(i)}^{(n+1)}\right) - \hat{f}\left(\varepsilon_{(i)}^{(n)}\right) \right\} \\
&= \frac{\kappa}{4} \sum_{i=1}^N \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(n+1)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right)} \right\} \\
&\quad \times \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \left(X_{(i+1)}^{(n+1)} - X_{(i)}^{(n)} \right) \\
&- \frac{\kappa}{4} \sum_{i=1}^N \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(n+1)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right)} \right\} \\
&\quad \times \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \left(X_{(i)}^{(n+1)} - X_{(i)}^{(n)} \right).
\end{aligned}$$

Since $X_{(N+1)}^{(n)} = X_{(1)}^{(n)}$ for any $n = 0, 1, \dots, K-1$, we have

$$\begin{aligned}
& l_{N*} \sum_{i=1}^N \left\{ \hat{f} \left(\varepsilon_{(i)}^{(n+1)} \right) - \hat{f} \left(\varepsilon_{(i)}^{(n)} \right) \right\} \\
&= \frac{\kappa}{4} \sum_{i=1}^N \left\{ \left\{ \varepsilon_{(i-1)}^{(n+1)} + \varepsilon_{(i-1)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i-1)}^{(n+1)}\right) \left(1 + \varepsilon_{(i-1)}^{(n)}\right)} \right\} \frac{X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left|X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)}\right| + \left|X_{(i)}^{(n)} - X_{(i-1)}^{(n)}\right|} \right. \\
&\quad \left. - \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(n+1)}\right) \left(1 + \varepsilon_{(i)}^{(n)}\right)} \right\} \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left|X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)}\right| + \left|X_{(i+1)}^{(n)} - X_{(i)}^{(n)}\right|} \right\} \\
&\quad \times \left(X_{(i)}^{(n+1)} - X_{(i)}^{(n)} \right).
\end{aligned}$$

By using (1.3.5), we have

$$l_{N*} \sum_{i=1}^N \left\{ \hat{f} \left(\varepsilon_{(i)}^{(n+1)} \right) - \hat{f} \left(\varepsilon_{(i)}^{(n)} \right) \right\} = -m \sum_{i=1}^N \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \left(X_{(i)}^{(n+1)} - X_{(i)}^{(n)} \right). \quad (2.3.4)$$

From (2.3.3) and (2.3.4), we obtain

$$\begin{aligned}
& F \left(X^{(n+1)}, V^{(n+1)} \right) - F \left(X^{(n)}, V^{(n)} \right) \\
&= m \sum_{i=1}^N \left(V_{(i)}^{(n+1)} - V_{(i)}^{(n)} \right) \frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t} - m \sum_{i=1}^N \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \left(X_{(i)}^{(n+1)} - X_{(i)}^{(n)} \right) \\
&= 0.
\end{aligned}$$

It means that (2.2.6) is the preserved energy for scheme (NS). \square

The second lemma gives the estimates for $X_{(i)}^{(n)}$ and $V_{(i)}^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}, i = 1, 2, \dots, N$.

Lemma 2.6. *Let $K \in \mathbb{Z}_{>0}$ and $(X^{(n)}, V^{(n)})$ be a solution of (NS) $(\Delta t, X^{(n-1)}, V^{(n-1)})$ for $n = 1, 2, \dots, K$, where $\Delta t = \frac{T}{K}$. If $(X^{(0)}, V^{(0)}) \in \mathbb{R}^{4N}$ and $X_{(i)}^{(0)} \neq X_{(i+1)}^{(0)}$ for $i = 1, 2, \dots, N$, then the following inequalities hold*

$$\left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right| \geq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}}, \quad (2.3.5)$$

$$\left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right| \leq \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}), \quad (2.3.6)$$

$$\left| V_{(i)}^{(n)} \right| \leq \sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})}, \quad (2.3.7)$$

for $i = 1, 2, \dots, N, n = 1, 2, \dots, K$, where $d_0 = F(X^{(0)}, V^{(0)})$, $X^{(n)} = (X_{(1)}^{(n)}, X_{(2)}^{(n)}, \dots, X_{(N)}^{(n)})$, $V^{(n)} = (V_{(1)}^{(n)}, V_{(2)}^{(n)}, \dots, V_{(N)}^{(n)})$ for $n = 0, 1, \dots, K$.

Proof. By Lemma 2.5 we have $F(X^{(n)}, V^{(n)}) = d_0$ for $n = 0, 1, \dots, K-1$, and then Lemma 2.2 guarantees (2.3.5), (2.3.6) and (2.3.7). Thus, we have proved this lemma. \square

The last lemma in this section is concerned with existence of a set on which $\widehat{\mathcal{F}}_K$ is a contraction mapping.

Lemma 2.7. *For $\delta > 0$, $M' > 0$, $M > 0$, put*

$$\widehat{W}(\delta, M', M) = \{z \in \mathbb{R}^{4N} \mid \delta \leq |X_{(i+1)} - X_{(i)}| \leq M', |V_{(i+1)} - V_{(i)}| \leq M \text{ for } i = 1, 2, \dots, N\}$$

Then there exists $K_0 \in \mathbb{Z}_{>0}$ such that for $K \geq K_0$, $\widehat{\mathcal{F}}_K : \widehat{W}(\delta, M', M) \rightarrow \widehat{W}(\delta, M', M)$ is a contraction mapping.

Proof. First, we prove that $\widehat{\mathcal{F}}_K$ is a function from $\widehat{W}(\delta, M', M)$ to $\widehat{W}(\delta, M', M)$. Let $K \in \mathbb{Z}_{>0}$, $\Delta t = \frac{T}{K}$, $(X, V) \in \widehat{W}(\delta, M', M)$ and put $\widehat{\mathcal{F}}_K((X, V)) = (X', V')$. We shall prove $(X', V') \in \widehat{W}(\delta, M', M)$ for some $K \in \mathbb{Z}_{>0}$, $\delta > 0$, $M' > 0$ and $M > 0$. By using (2.3.2), (2.3.5) and (2.3.7), we have

$$\begin{aligned} |X'_{(i+1)} - X'_{(i)}| &= |\hat{f}_{K(i+1)}(V, V^{(n)}, X^{(n)}) - \hat{f}_{K(i)}(V, V^{(n)}, X^{(n)})| \\ &\geq |X^{(n)}_{(i+1)} - X^{(n)}_{(i)}| - \frac{\Delta t}{2} (|V_{(i+1)} - V_{(i)}| + |V^{(n)}_{(i+1)}| + |V^{(n)}_{(i)}|) \\ &\geq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} - \frac{\Delta t}{2} \left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right) \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

In the calculation above, since $(X, V) \in \widehat{W}(\delta, M', M)$, the third inequality holds. Here, we choose $\delta > 0$ such that $2\delta \leq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}}$, then we have

$$|X'_{(i+1)} - X'_{(i)}| \geq 2\delta - \frac{\Delta t}{2} \left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right) \quad \text{for } i = 1, 2, \dots, N.$$

Moreover, we choose $K \in \mathbb{Z}_{>0}$ such that

$$\frac{\Delta t}{2} \left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right) \leq \delta, \quad (2.3.8)$$

then we have

$$|X'_{(i+1)} - X'_{(i)}| \geq \delta \quad \text{for } i = 1, 2, \dots, N.$$

In the same way, by using (2.3.6) and (2.3.7), we have

$$\begin{aligned} |X'_{(i+1)} - X'_{(i)}| &\leq \frac{m\Delta t}{2} (|V_{(i+1)} - V_{(i)}| + |V^{(n)}_{(i+1)}| + |V^{(n)}_{(i)}|) + |X^{(n)}_{(i+1)} - X^{(n)}_{(i)}| \\ &\leq \frac{m\Delta t}{2} \left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right) + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

Here, we choose $M' > 0$ such that $\frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) \leq \frac{M'}{2}$. Moreover, we take $\Delta t > 0$ such that

$$\frac{m\Delta t}{2} \left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right) \leq \frac{M'}{2}, \quad (2.3.9)$$

we have

$$\left| X'_{(i+1)} - X'_{(i)} \right| \leq \frac{M'}{2} + \frac{M'}{2} = M' \quad \text{for } i = 1, 2, \dots, N.$$

Next, we give an estimate for $V'_{(i)}$ for $i = 1, 2, \dots, N$. By using (2.3.1), we have

$$\begin{aligned} \left| V'_{(i+1)} - V'_{(i)} \right| &= \left| g_{(i+1)} \left(X, X^{(n)}, V^{(n)} \right) - g_{(i)} \left(X, X^{(n)}, V^{(n)} \right) \right| \\ &\leq \frac{\kappa\Delta t}{4} \left| \frac{(\varepsilon_{(i)} + \varepsilon_{(i)}^{(n)} + 1) \left(X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right)}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{1}{(1 + \varepsilon_{(i)}) (1 + \varepsilon_{(i)}^{(n)})} \frac{X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{(\varepsilon_{(i+1)} + \varepsilon_{(i+1)}^{(n)} + 1) \left(X_{(i+2)} - X_{(i+1)} + X_{(i+2)}^{(n)} - X_{(i+1)}^{(n)} \right)}{|X_{(i+2)} - X_{(i+1)}| + |X_{(i+2)}^{(n)} - X_{(i+1)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{1}{(\varepsilon_{(i+1)} + 1) (\varepsilon_{(i+1)}^{(n)} + 1)} \frac{X_{(i+2)} - X_{(i+1)} + X_{(i+2)}^{(n)} - X_{(i+1)}^{(n)}}{|X_{(i+2)} - X_{(i+1)}| + |X_{(i+2)}^{(n)} - X_{(i+1)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{(\varepsilon_{(i-1)} + \varepsilon_{(i-1)}^{(n)} + 1) \left(X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right)}{|X_{(i)} - X_{(i-1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{1}{(1 + \varepsilon_{(i-1)}) (1 + \varepsilon_{(i-1)}^{(n)})} \frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{|X_{(i)} - X_{(i-1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{(\varepsilon_{(i)} + \varepsilon_{(i)}^{(n)} + 1) \left(X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right)}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\ &\quad + \frac{\kappa\Delta t}{4} \left| \frac{1}{(\varepsilon_{(i)} + 1) (\varepsilon_{(i)}^{(n)} + 1)} \frac{X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\ &\quad + \left| V_{(i+1)}^{(n)} - V_{(i)}^{(n)} \right| \\ &=: \sum_{j=1}^9 D_j \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

Since the estimates for D_1 , D_3 , D_5 and D_7 can be obtained in the same way, so we shall estimate

D_1 , here. By using (2.3.6) and $(X, V) \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
|D_1| &= \frac{\kappa \Delta t}{4} \left| \frac{(\varepsilon_{(i)} + \varepsilon_{(i)}^{(n)} + 1) (X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)})}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\
&\leq \frac{\kappa \Delta t}{4l_{N*}} \left(|l_{(i)}| + |l_{(i)}^{(n)}| + l_{N*} \right) \\
&\leq \frac{\kappa \Delta t}{4l_{N*}} \left(|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}| + l_{N*} \right) \\
&\leq \frac{\kappa \Delta t}{4l_{N*}} \left(M' + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) + l_{N*} \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|D_3| &\leq \frac{\kappa \Delta t}{4l_{N*}} \left(M' + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) + l_{N*} \right), \\
|D_5| &\leq \frac{\kappa \Delta t}{4l_{N*}} \left(M' + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) + l_{N*} \right), \\
|D_7| &\leq \frac{\kappa \Delta t}{4l_{N*}} \left(M' + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) + l_{N*} \right).
\end{aligned}$$

Next, since the estimates for D_2 , D_4 , D_6 and D_8 can be obtained in the same way, so we shall estimate D_2 , here. By using (2.3.5) and $(X, V) \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
|D_2| &= \frac{\kappa \Delta t}{4} \left| \frac{1}{(1 + \varepsilon_{(i)}) (1 + \varepsilon_{(i)}^{(n)})} \frac{X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{|X_{(i+1)} - X_{(i)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right| \\
&\leq \frac{\kappa l_{N*} \Delta t}{4} \left(\frac{1}{|l_{(i)}|} \frac{1}{|l_{(i)}^{(n)}|} \right) \\
&= \frac{\kappa l_{N*}^2 \Delta t}{4} \left(\frac{1}{|X_{(i+1)} - X_{(i)}|} \frac{1}{|X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \right) \\
&\leq \Delta t \frac{(4d_0 + N\kappa l_{N*})}{4\delta}.
\end{aligned}$$

Similarly, we have

$$|D_4| \leq \Delta t \frac{(4d_0 + N\kappa l_{N*})}{4\delta}, \quad |D_6| \leq \Delta t \frac{(4d_0 + N\kappa l_{N*})}{4\delta}, \quad |D_8| \leq \Delta t \frac{(4d_0 + N\kappa l_{N*})}{4\delta}.$$

Finally, by using (2.3.7), we have

$$|D_9| \leq 2\sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})}.$$

Therefore, we choose $M > 0$ and $\Delta t > 0$ such that $2\sqrt{\frac{1}{2m} (4d_0 + N\kappa l_{N*})} \leq \frac{M}{9}$ and

$$\Delta t = \min \left\{ \frac{4l_{N*}M}{9\kappa \left\{ M' + \frac{1}{\kappa} (4d_0 + N\kappa l_{N*}) + l_{N*} \right\}}, \quad \frac{4\delta M}{9(4d_0 + N\kappa l_{N*}) + l_{N*}} \right\}, \quad (2.3.10)$$

then we have

$$\left| V'_{(i+1)} - V'_{(i)} \right| \leq \sum_{j=1}^9 |D_j| \leq M \quad \text{for } i = 1, 2, \dots, N.$$

By combining (2.3.8), (2.3.9) and (2.3.10), we take $\Delta t > 0$ as follows, again

$$\Delta t = \min \left\{ \frac{2\delta}{m \left(M + 2\sqrt{\frac{1}{m}(4d_0 + N\kappa l_{N*})} \right)}, \frac{2M'}{m \left\{ M + \sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right\} + \frac{1}{\kappa}(4d_0 + N\kappa l_{N*})}, \right. \\ \left. \frac{4l_{N*}M}{9\kappa \left\{ M' + \frac{1}{\kappa}(4d_0 + N\kappa l_{N*}) + l_{N*} \right\}}, \frac{4\delta M}{9(4d_0 + N\kappa l_{N*}) + l_{N*}} \right\},$$

and then we see that $(X', V') \in \widehat{W}(\delta, M', M)$. Thus, we have proved that $\widehat{\mathcal{F}}_K$ is a function from $\widehat{W}(\delta, M', M)$ to $\widehat{W}(\delta, M', M)$.

Finally, we shall show that $\widehat{\mathcal{F}}_K: \widehat{W}(\delta, M', M) \rightarrow \widehat{W}(\delta, M', M)$ is a contraction mapping. Let $Z, Z' \in \widehat{W}(\delta, M', M)$ and put $Z = (X, V)$, $Z' = (X', V')$. Easily, we have

$$\begin{aligned} \left\| \widehat{F}_K(Z) - \widehat{F}_K(Z') \right\|_{\mathbb{R}^{4N}}^2 &= \sum_{i=1}^N \left| \widehat{f}_{K(i)}(V, V^{(n)}, X^{(n)}) - \widehat{f}_{K(i)}(V', V^{(n)}, X^{(n)}) \right|^2 \\ &\quad + \sum_{i=1}^N \left| \widehat{g}_{K(i)}(X, X^{(n)}, V^{(n)}) - \widehat{g}_{K(i)}(X', X^{(n)}, V^{(n)}) \right|^2 \\ &\leq \sum_{i=1}^N \left| \widehat{f}_{K(i)}(V, V^{(n)}, X^{(n)}) - \widehat{f}_{K(i)}(V', V^{(n)}, X^{(n)}) \right| \\ &\quad + \sum_{i=1}^N \left| \widehat{g}_{K(i)}(X, X^{(n)}, V^{(n)}) - \widehat{g}_{K(i)}(X', X^{(n)}, V^{(n)}) \right|. \end{aligned}$$

Here, by using (2.3.2), we have

$$\begin{aligned} \left| \widehat{f}_{K(i)}(V, V^{(n)}, X^{(n)}) - \widehat{f}_{K(i)}(V', V^{(n)}, X^{(n)}) \right| &= \frac{\Delta t}{2} |V_{(i)} - V'_{(i)}| \\ &\leq \frac{\Delta t}{2} \|Z - Z'\|_{\mathbb{R}^{4N}} \quad \text{for } i = 1, 2, \dots, N. \end{aligned} \quad (2.3.11)$$

Next, by using (2.3.1), we have

$$\begin{aligned} &\left| \widehat{g}_{K(i)}(X, X^{(n)}, V^{(n)}) - \widehat{g}_{K(i)}(X', X^{(n)}, V^{(n)}) \right| \\ &\leq \frac{\kappa \Delta t}{4m} \left| \frac{(\varepsilon_{(i-1)} + \varepsilon_{(i-1)}^{(n)}) (X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)})}{|X_{(i)} - X_{(i-1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \right. \\ &\quad \left. - \frac{(\varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)}) (X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)})}{|X'_{(i)} - X'_{(i-1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{\kappa \Delta t}{4m} \left| \frac{\left(\varepsilon_{(i)} + \varepsilon_{(i)}^{(n)} \right) \left(X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right)}{\left| X_{(i+1)} - X_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \right. \\
& \quad \left. - \frac{\left(\varepsilon'_{(i)} + \varepsilon_{(i)}^{(n)} \right) \left(X'_{(i+1)} - X'_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right)}{\left| X'_{(i+1)} - X'_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \right| \\
& + \frac{\kappa \Delta t}{4m} \left| \frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} - \frac{X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right| \\
& + \frac{\kappa \Delta t}{4m} \left| \frac{X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X_{(i+1)} - X_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} - \frac{X'_{(i+1)} - X'_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X'_{(i+1)} - X'_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \right| \\
& + \frac{\kappa \Delta t}{4m} \left| \frac{1}{\left(1 + \varepsilon_{(i-1)} \right) \left(1 + \varepsilon_{(i-1)}^{(n)} \right)} \frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right. \\
& \quad \left. - \frac{1}{\left(1 + \varepsilon'_{(i-1)} \right) \left(1 + \varepsilon_{(i-1)}^{(n)} \right)} \frac{X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right| \\
& + \frac{\kappa \Delta t}{4m} \left| \frac{1}{\left(1 + \varepsilon_{(i)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right)} \frac{X_{(i+1)} - X_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X_{(i+1)} - X_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \right. \\
& \quad \left. - \frac{1}{\left(1 + \varepsilon'_{(i)} \right) \left(1 + \varepsilon_{(i)}^{(n)} \right)} \frac{X'_{(i+1)} - X'_{(i)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{\left| X'_{(i+1)} - X'_{(i)} \right| + \left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \right| \\
& =: E_1 + E_2 + E_3 + E_4 + E_5 + E_6 \quad \text{for } i = 1, 2, \dots, N.
\end{aligned}$$

For E_1 we have

$$\begin{aligned}
& E_1 \\
& \leq \frac{\kappa \Delta t}{4m} \frac{1}{H} \left\{ \left| \left(\varepsilon_{(i-1)} - \varepsilon'_{(i-1)} \right) \left(X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \right| \right. \\
& \quad + \left| \left(\varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right) \left(X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| - \left| X_{(i)} - X_{(i-1)} \right| \right) \right| \\
& \quad \left. + \left| \left(\varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right) \left(X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)} \right) \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \right| \right\} \\
& =: \frac{\kappa \Delta t}{4m} \frac{1}{H} (E_{1,1} + E_{1,2} + E_{1,3}),
\end{aligned}$$

where $H = \frac{1}{\left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)}$. Immediately,

we have

$$\begin{aligned}
\frac{E_{1,1}}{H} &= \frac{\left| \left(\varepsilon_{(i-1)} - \varepsilon'_{(i-1)} \right) \left(X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right) \right|}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \\
&\leq \frac{\left| \left(\varepsilon_{(i-1)} - \varepsilon'_{(i-1)} \right) \right| \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \\
&\leq \frac{1}{l_{N*}} \left| X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)} \right| \\
&\leq \frac{1}{l_{N*}} \left(\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right| \right) \\
&\leq \frac{2}{l_{N*}} \|Z - Z'\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

Since it holds

$$\begin{aligned}
E_{1,2} &= \left| \left(\varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right) \left(X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| - \left| X_{(i)} - X_{(i-1)} \right| \right) \right| \\
&\leq \left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right| \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \left(\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right| \right) \\
&\leq 2 \left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right| \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \|Z - Z'\|_{\mathbb{R}^{4N}},
\end{aligned}$$

we have

$$\frac{E_{1,2}}{H} \leq \frac{2 \left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right|}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Here, it is easy to see that

$$\left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right| \leq \frac{1}{l_{N*}} \left(\left| l'_{(i-1)} \right| + \left| l_{(i-1)}^{(n)} \right| + 2l_{N*} \right) \quad \text{for } i = 1, 2, \dots, N,$$

$$\frac{\left| l'_{(i-1)} \right|}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} < 1, \quad \frac{\left| l_{(i-1)}^{(n)} \right|}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} < 1 \quad \text{for } i = 1, 2, \dots, N.$$

Moreover, by using (2.3.5) in Lemma 2.6, we see that

$$\frac{1}{\left| X_{(i+1)}^{(n)} - X_{(i)}^{(n)} \right|} \leq \frac{4d_0 + N\kappa l_{N*}}{\kappa l_{N*}^2} \quad \text{for } i = 1, 2, \dots, N. \quad (2.3.12)$$

From these inequalities, we have

$$\frac{E_{1,2}}{H} \leq 4 \left(\frac{1}{l_{N*}} + \frac{4d_0 + N\kappa l_{N*}}{\kappa l_{N*}^2} \right) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Next, we have

$$\begin{aligned}
E_{1,3} &= \left| \left(\varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right) \left(X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)} \right) \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \right| \\
&\leq \left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right| \left(\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right| \right) \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \\
&\leq 2 \left| \varepsilon'_{(i-1)} + \varepsilon_{(i-1)}^{(n)} \right| \left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \|Z - Z'\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

In the same way for $\frac{E_{1,2}}{H}$, we have

$$\frac{E_{1,3}}{H} \leq 4 \left(\frac{1}{l_{N*}} + \frac{4d_0 + N\kappa l_{N*}}{\kappa l_{N*}^2} \right) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Therefore, we obtain

$$E_1 \leq \frac{\kappa \Delta t}{2m} \left\{ \frac{5}{l_{N*}} + \frac{4}{\kappa l_{N*}^2} (4d_0 + N\kappa l_{N*}) \right\} \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Similarly, we have

$$E_2 \leq \frac{\kappa \Delta t}{2m} \left\{ \frac{5}{l_{N*}} + \frac{4}{\kappa l_{N*}^2} (4d_0 + N\kappa l_{N*}) \right\} \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Next, we have

$$\begin{aligned}
E_3 &= \frac{\kappa \Delta t}{4m} \left| \frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} - \frac{X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right| \\
&\leq \frac{\kappa \Delta t}{4m} \frac{1}{H} \left\{ \left| \left(X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)} \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \right| \right. \\
&\quad \left. + \left| \left(X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)} - X_{(i-1)} \right| \right) \right| \right\} \\
&=: \frac{\kappa \Delta t}{4m} \frac{1}{H} (E_{3,1} + E_{3,2}).
\end{aligned}$$

Since

$$\begin{aligned}
E_{3,1} &\leq \left(\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right| \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \\
&\leq 2 \|Z - Z'\| \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{E_{3,1}}{H} &\leq = \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}} \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)}{\left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)} \\
&= \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|}
\end{aligned}$$

$$\leq \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|}.$$

In the same way for $E_{3,1}$, we have

$$\frac{E_{3,2}}{H} \leq \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|}.$$

Therefore, we have

$$E_3 \leq \frac{\kappa \Delta t}{m} \frac{\|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|}.$$

By using (2.3.12), we have

$$E_3 \leq \frac{\Delta t}{m l_{N*}^2} (4d_0 + N \kappa l_{N*}) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

In the same way for E_3 , we have

$$E_4 \leq \frac{\Delta t}{m l_{N*}^2} (4d_0 + N \kappa l_{N*}) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Next, we have

$$\begin{aligned} E_5 &\leq \frac{\kappa \Delta t}{4m} \left| \frac{1}{1 + \varepsilon_{(i-1)}^{(n)}} \left(\frac{1}{1 + \varepsilon_{(i-1)}} - \frac{1}{1 + \varepsilon'_{(i-1)}} \right) \left(\frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right) \right| \\ &\quad + \frac{\kappa \Delta t}{4m} \left| \frac{1}{1 + \varepsilon_{(i-1)}^{(n)}} \frac{1}{1 + \varepsilon'_{(i-1)}} \left(\frac{X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right) \right| \\ &\quad + \frac{\kappa \Delta t}{4m} \left| \frac{(X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)})}{(1 + \varepsilon_{(i-1)}^{(n)})(1 + \varepsilon'_{(i-1)})} \right. \\ &\quad \times \left. \left\{ \frac{\left| X'_{(i)} - X'_{(i-1)} \right| - \left| X_{(i)} - X_{(i-1)} \right|}{\left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)} \right\} \right| \\ &=: \frac{\kappa \Delta t}{4m} (E_{5,1} + E_{5,2} + E_{5,3}). \end{aligned}$$

We have

$$\begin{aligned} E_{5,1} &= \left| \frac{1}{1 + \varepsilon_{(i-1)}^{(n)}} \frac{\varepsilon'_{(i-1)} - \varepsilon_{(i-1)}}{(1 + \varepsilon_{(i-1)})(1 + \varepsilon'_{(i-1)})} \left(\frac{X_{(i)} - X_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right) \right| \\ &\leq \left| \frac{l_{N*}}{l_{(i-1)}^{(n)}} \left(\frac{l_{N*}}{l_{(i-1)}} \right) \left(\frac{l_{N*}}{l'_{(i-1)}} \right) \frac{l'_{(i-1)} - l_{(i-1)}}{l_{N*}} \right| \end{aligned}$$

$$\begin{aligned}
&= l_{N*}^2 \left| \frac{1}{l_{(i-1)}^{(n)}} \left(\frac{l'_{(i-1)} - l_{(i-1)}}{l_{(i-1)} l'_{(i-1)}} \right) \right| \\
&\leq \frac{l_{N*}^2 \left(|X_{(i)} - X'_{(i)}| + |X_{(i-1)} - X'_{(i-1)}| \right)}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \left| X_{(i)} - X_{(i-1)} \right| \left| X'_{(i)} - X'_{(i-1)} \right|} \\
&\leq \frac{2l_{N*}^2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \left| X_{(i)} - X_{(i-1)} \right| \left| X'_{(i)} - X'_{(i-1)} \right|}.
\end{aligned}$$

By using (2.3.12), we have

$$E_{5,1} \leq \frac{2(4d_0 + N\kappa l_{N*})}{\kappa} \frac{\|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)} - X_{(i-1)} \right| \left| X'_{(i)} - X'_{(i-1)} \right|}.$$

Since $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$E_{5,1} \leq \frac{2(4d_0 + N\kappa l_{N*})}{\kappa \delta^2} \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

In the same way for $E_{5,1}$, by using (2.3.12) and $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
E_{5,2} &= \left| \frac{1}{1 + \varepsilon_{(i-1)}^{(n)}} \frac{1}{1 + \varepsilon'_{(i-1)}} \left(\frac{X_{(i)} - X_{(i-1)} - X'_{(i)} + X'_{(i-1)}}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right) \right| \\
&\leq l_{N*}^2 \left| \frac{1}{l_{(i-1)}^{(n)} l'_{(i-1)}} \right| \left(\frac{\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right|}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \right) \\
&\leq \frac{l_{N*}^2}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \left| X'_{(i)} - X'_{(i-1)} \right|} \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)} - X_{(i-1)} \right|} \\
&\leq \frac{2(4d_0 + N\kappa l_{N*})}{\kappa \delta^2} \|Z - Z'\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

Finally, in the same way for $E_{5,1}$ and $E_{5,2}$, by using (2.3.12) and $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
E_{5,3} &= \left| \frac{\left(X'_{(i)} - X'_{(i-1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right)}{\left(1 + \varepsilon_{(i-1)}^{(n)} \right) \left(1 + \varepsilon'_{(i-1)} \right)} \right. \\
&\quad \times \left. \left\{ \frac{\left| X'_{(i)} - X'_{(i-1)} \right| - \left| X_{(i)} - X_{(i-1)} \right|}{\left(\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right) \left(\left| X'_{(i)} - X'_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \right)} \right\} \right| \\
&\leq l_{N*}^2 \left| \frac{1}{l_{(i-1)}^{(n)} l'_{(i-1)}} \right| \frac{\left| X_{(i)} - X'_{(i)} \right| + \left| X_{(i-1)} - X'_{(i-1)} \right|}{\left| X_{(i)} - X_{(i-1)} \right| + \left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right|} \\
&\leq \frac{l_{N*}^2}{\left| X_{(i)}^{(n)} - X_{(i-1)}^{(n)} \right| \left| X'_{(i)} - X'_{(i-1)} \right|} \frac{2 \|Z - Z'\|_{\mathbb{R}^{4N}}}{\left| X_{(i)} - X_{(i-1)} \right|}
\end{aligned}$$

$$\leq \frac{2(4d_0 + N\kappa l_{N*})}{\kappa\delta^2} \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Accordingly, we obtain

$$E_5 \leq \frac{3\Delta t}{2m\delta^2} (4d_0 + N\kappa l_{N*}) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

In the same way for E_5 , we obtain

$$E_6 \leq \frac{3\Delta t}{2m\delta^2} (4d_0 + N\kappa l_{N*}) \|Z - Z'\|_{\mathbb{R}^{4N}}.$$

Therefore, we obtain

$$\begin{aligned} & \left| \hat{g}_{Ki} \left(X, X^{(n)}, V^{(n)} \right) - \hat{g}_{Ki} \left(X', X^{(n)}, V^{(n)} \right) \right| \\ & \leq \frac{\kappa\Delta t}{m} \left\{ \frac{5\kappa}{l_{N*}} + \frac{3}{\kappa} \left(\frac{2}{l_{N*}^2} + \frac{1}{\delta^2} \right) (4d_0 + N\kappa l_{N*}) \right\} \|Z - Z'\|_{\mathbb{R}^{4N}} \quad \text{for } i = 1, 2, \dots, N. \end{aligned}$$

By using the above inequality and (2.3.11), we obtain

$$\begin{aligned} & \left\| \hat{\mathcal{F}}_K(Z) - \hat{\mathcal{F}}_K(Z') \right\|_{\mathbb{R}^{4N}} \\ & \leq \sum_{i=1}^N \frac{\Delta t}{2} \|Z - Z'\|_{\mathbb{R}^{4N}} + \sum_{i=1}^N \frac{\kappa\Delta t}{m} \left\{ \frac{5\kappa}{l_{N*}} + \frac{3}{\kappa} \left(\frac{2}{l_{N*}^2} + \frac{1}{\delta^2} \right) (4d_0 + N\kappa l_{N*}) \right\} \|Z - Z'\|_{\mathbb{R}^{4N}} \\ & \leq \frac{N\Delta t}{2m} \left\{ m + 2\kappa \left\{ \frac{5\kappa}{l_{N*}} + \frac{3}{\kappa} \left(\frac{2}{l_{N*}^2} + \frac{1}{\delta^2} \right) (4d_0 + N\kappa l_{N*}) \right\} \right\} \|Z - Z'\|_{\mathbb{R}^{4N}} \\ & \quad \text{for } Z, Z' \in \widehat{W}(\delta, M', M). \end{aligned}$$

Hence, by choosing $\Delta t > 0$ such that

$$\frac{N\Delta t}{2m} \left\{ m + 2\kappa \left\{ \frac{5\kappa}{l_{N*}} + \frac{3}{\kappa} \left(\frac{2}{l_{N*}^2} + \frac{1}{\delta^2} \right) (4d_0 + N\kappa l_{N*}) \right\} \right\} < 1,$$

$\hat{\mathcal{F}}_K : \widehat{W}(\delta, M', M) \rightarrow \widehat{W}(\delta, M', M)$ is a contraction. \square

Proof of Theorem 2.2. By Lemma 2.7, there exists $K_0 \in \mathbb{Z}_{>0}$ such that for $K \geq K_0$ $\hat{\mathcal{F}}_K : \widehat{W}(\delta, M', M) \rightarrow \widehat{W}(\delta, M', M)$ is a contraction. From the Banach fixed point theorem, $\hat{\mathcal{F}}_K$ has a unique fixed point \hat{Z} for $K \geq K_0$. Consequently, Theorem 2.2 has been proved. \square

2.4 Error estimate

In this section, we prove Theorem 2.3. For its proof we need several steps. Let $K \geq K_0$, $\Delta t = \frac{T}{K}$ and $(X^{(n)}, V^{(n)}) \in \mathbb{R}^4$ be a solution of (NS) $(\Delta t, X^{(n-1)}, V^{(n-1)})$ for $n = 1, 2, \dots, K$. Also, we define the approximate solution $X^{(K)} : [0, T] \rightarrow \mathbb{R}^{2N}$ and the approximation of the derivative $V^{(K)} : [0, T] \rightarrow \mathbb{R}^{2N}$ of (OP) $(X^{(0)}, V^{(0)})$ in the following way

$$X_{(i)}^{(K)}(t) = \frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t} (t - n\Delta t) + X_{(i)}^{(n)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \dots, K-1,$$

$$V_{(i)}^{(K)}(t) = \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} (t - n\Delta t) + V_{(i)}^{(n)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \dots, K-1, \quad (2.4.1)$$

$$\text{and } X^{(K)} = \left(X_{(1)}^{(K)}, X_{(2)}^{(K)}, \dots, X_{(N)}^{(K)} \right), \quad V^{(K)} = \left(V_{(1)}^{(K)}, V_{(2)}^{(K)}, \dots, V_{(N)}^{(K)} \right).$$

Now, we see that $X_{(i)}^{(K)}, V_{(i)}^{(K)} \in W^{1,2}(0, T)^2$ for $i = 1, 2, \dots, N$ and

$$\begin{aligned} \frac{dX_{(i)}^{(K)}}{dt} &= \sum_{n=0}^{K-1} \frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t} \chi_{[n\Delta t, (n+1)\Delta t]}, \\ \frac{dV_{(i)}^{(K)}}{dt} &= \sum_{n=0}^{K-1} \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \chi_{[n\Delta t, (n+1)\Delta t]} \quad \text{for } i = 1, 2, \dots, N, \end{aligned} \quad (2.4.2)$$

where χ_I is the characteristic function for the interval $I \subset [0, T]$.

The first lemma is concerned with the uniform estimates for the derivative of the approximation solution.

Lemma 2.8. *There exists $M_1 > 0$ such that*

$$\left| \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \right| \leq M_1, \quad |V_{(i)}^{(n)}| \leq M_1,$$

for any $K \in \mathbb{Z}_{>0}$ with $K \geq K_0$, $i = 1, 2, \dots, N$ and $n = 0, 1, \dots, N-1$.

Proof. Let $K \in \mathbb{Z}_{>0}$ with $K \geq K_0$, $n = 0, 1, \dots, K-1$ and $i = 1, 2, \dots, N$. By using (1.3.5), we have

$$\begin{aligned} \left| \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \right| &\leq \frac{\kappa}{4m} \left\{ \left| \varepsilon_{(i-1)}^{(n+1)} \right| + \left| \varepsilon_{(i-1)}^{(n)} \right| + 1 + \frac{1}{\left| 1 + \varepsilon_{(i-1)}^{(n+1)} \right| \left| 1 + \varepsilon_{(i-1)}^{(n)} \right|} \right\} \\ &\quad + \frac{\kappa}{4m} \left\{ \left| \varepsilon_{(i)}^{(n+1)} \right| + \left| \varepsilon_{(i)}^{(n)} \right| + 1 + \frac{1}{\left| 1 + \varepsilon_{(i)}^{(n+1)} \right| \left| 1 + \varepsilon_{(i)}^{(n)} \right|} \right\}. \end{aligned}$$

By using (2.3.5) and (2.3.6) in Lemma 2.6, we have

$$\left| \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \right| \leq \frac{\kappa}{2m} \left\{ \frac{2(4d_0 + N\kappa l_{N*})}{\kappa l_{N*}} + 3 + \frac{(4d_0 + N\kappa l_{N*})^2}{\kappa^2 l_{N*}^2} \right\}.$$

We put $M_1 > 0$ by

$$M_1 = \max \left\{ \frac{\kappa}{2m} \left\{ \frac{2(4d_0 + N\kappa l_{N*})}{\kappa l_{N*}} + 3 + \frac{(4d_0 + N\kappa l_{N*})^2}{\kappa^2 l_{N*}^2} \right\}, \sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \right\},$$

and then Lemma 2.6 implies

$$\left| \frac{V_{(i)}^{(n+1)} - V_{(i)}^{(n)}}{\Delta t} \right| \leq M_1, \quad |V_{(i)}^{(n)}| \leq M_1 \quad \text{for } K \geq K_0, n = 0, 1, \dots, K-1 \text{ and } i = 1, 2, \dots, N.$$

This guarantees that Lemma 2.8 holds. \square

Now, we put

$$\xi_{(i)}^{(K)}(t) = X_{(i)}^{(K)}(t) - X_{(i)}^{(K)}(0) - \int_0^t V_{(i)}^{(K)}(\tau) d\tau, \quad (2.4.3)$$

for $t \in [n\Delta t, (n+1)\Delta t]$, $n = 0, 1, \dots, K-1$, $K \geq K_0$ and $i = 1, 2, \dots, N$. The second lemma shows the convergence of $\xi_{(i)}^{(K)}$ as $K \rightarrow \infty$ for $i = 1, 2, \dots, N$.

Lemma 2.9. *The following uniform estimate holds*

$$\left| \xi_{(i)}^{(K)}(t) \right| \leq 2M_1(\Delta t)^2 \quad \text{for } K \geq K_0 \text{ and } i = 1, 2, \dots, N,$$

where M_1 is the same constant as in Lemma 2.8, namely, $\xi_{(i)}^{(K)} \rightarrow 0$ in $C([0, T])^2$ as $K \rightarrow \infty$ for $i = 1, 2, \dots, N$.

Proof. Put $X_{(i)}^{(K)} = (X_{(i)1}^{(K)}, X_{(i)2}^{(K)})$, $V_{(i)}^{(K)} = (V_{(i)1}^{(K)}, V_{(i)2}^{(K)})$ and $\xi_{(i)}^{(K)} = (\xi_{(i)1}^{(K)}, \xi_{(i)2}^{(K)})$ for $i = 1, 2, \dots, N$. By using (2.4.2) and (1.3.6), we have

$$\frac{dX_{(i)j}^{(K)}}{dt} = \sum_{n=0}^{K-1} \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} \chi_{[n\Delta t, (n+1)\Delta t)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t), n = 0, 1, \dots, K-1.$$

By integrating both sides of this equation, we have

$$\int_0^t \frac{dX_{(i)j}^{(K)}}{dt}(\tau) d\tau = \sum_{n=0}^{K-1} \int_0^t \frac{V_{(i)j}^{(n+1)}(\tau) + V_{(i)j}^{(n)}(\tau)}{2} \chi_{[n\Delta t, (n+1)\Delta t)}(\tau) d\tau, \\ \text{for } K \geq K_0, i = 1, 2, \dots, N, j = 1, 2 \text{ and } t \in [0, T].$$

For the left hand side of the equation, we see that

$$\int_0^t \frac{dX_{(i)j}^{(K)}}{dt}(\tau) d\tau = X_{(i)j}^{(K)}(t) - X_{(i)j}^{(K)}(0) \quad \text{for } K \geq K_0, i = 1, 2, \dots, N, j = 1, 2 \text{ and } t \in [0, T],$$

and for the right hand side, we have

$$\begin{aligned} & \int_0^t \sum_{n=0}^{K-1} \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} \chi_{[n\Delta t, (n+1)\Delta t)} d\tau \\ &= \int_0^{n\Delta t} \sum_{m=0}^{n-1} \frac{V_{(i)j}^{(m+1)} + V_{(i)j}^{(m)}}{2} \chi_{[m\Delta t, (m+1)\Delta t)} d\tau + \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} (t - n\Delta t) \\ &= \sum_{m=0}^{n-1} \frac{V_{(i)j}^{(m+1)} + V_{(i)j}^{(m)}}{2} \Delta t + \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} (t - n\Delta t), \end{aligned}$$

for $K \geq K_0, i = 1, 2, \dots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \dots, K-1$.

Therefore, we have

$$X_{(i)j}^{(K)}(t) - X_{(i)j}^{(K)}(0) = \sum_{m=0}^{n-1} \frac{V_{(i)j}^{(m+1)} + V_{(i)j}^{(m)}}{2} \Delta t + \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} (t - n\Delta t), \quad (2.4.4)$$

for $K \geq K_0, i = 1, 2, \dots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \dots, K-1$.

By using (2.4.1), we have

$$V_{(i)j}^{(K)} = \sum_{n=0}^{K-1} \left\{ \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} (t - n\Delta t) + V_{(i)j}^{(n)} \right\} \chi_{[n\Delta t, (n+1)\Delta t]},$$

for $K \geq K_0, i = 1, 2, \dots, N, j = 1, 2$ and $t \in [0, T]$.

By integrating both sides of the equation, we have

$$\begin{aligned} & \int_0^t V_{(i)j}^{(K)}(\tau) d\tau \\ &= \int_0^t \sum_{n=0}^{K-1} \left\{ \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} (\tau - n\Delta t) + V_{(i)j}^{(n)} \right\} \chi_{[n\Delta t, (n+1)\Delta t]} d\tau \\ &= \int_0^{n\Delta t} \sum_{m=0}^{n-1} \left\{ \frac{V_{(i)j}^{(m+1)} - V_{(i)j}^{(m)}}{\Delta t} (\tau - m\Delta t) + V_{(i)j}^{(m)} \right\} \chi_{[m\Delta t, (m+1)\Delta t]} d\tau \\ & \quad + \int_{n\Delta t}^t \left\{ \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} (\tau - n\Delta t) + V_{(i)j}^{(n)} \right\} \chi_{[n\Delta t, (n+1)\Delta t]} d\tau \\ &= \sum_{m=0}^{n-1} \left\{ \frac{V_{(i)j}^{(m+1)} - V_{(i)j}^{(m)}}{\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} (\tau - m\Delta t) d\tau + V_{(i)j}^{(m)} \Delta t \right\} \\ & \quad + \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \int_{n\Delta t}^t (\tau - n\Delta t) d\tau + V_{(i)j}^{(n)} (t - n\Delta t) \\ &= \sum_{m=0}^{n-1} \frac{V_{(i)j}^{(m+1)} + V_{(i)j}^{(m)}}{2} \Delta t + \left\{ \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \frac{1}{2} (t - n\Delta t) + V_{(i)j}^{(n)} \right\} (t - n\Delta t), \quad (2.4.5) \end{aligned}$$

for $t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \dots, K-1$.

By using (2.4.3), (2.4.4) and (2.4.5), we have

$$\begin{aligned} |\xi_{(i)j}^{(K)}(t)| &= \left| X_{(i)j}^{(K)}(t) - X_{(i)j}^{(K)}(0) - \int_0^t V_{(i)j}^{(K)} d\tau \right| \\ &= (t - n\Delta t) \left| \frac{V_{(i)j}^{(n+1)} + V_{(i)j}^{(n)}}{2} - \left\{ \left(\frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \frac{1}{2} (t - n\Delta t) \right) + V_{(i)j}^{(n)} \right\} \right| \end{aligned}$$

$$\begin{aligned}
&= (t - n\Delta t) \left| \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{2} - \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \frac{1}{2}(t - n\Delta t) \right| \\
&\leq (t - n\Delta t) \left(\frac{\Delta t}{2} \left| \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \right| + \frac{1}{2}(t - n\Delta t) \left| \frac{V_{(i)j}^{(n+1)} - V_{(i)j}^{(n)}}{\Delta t} \right| \right),
\end{aligned}$$

for $K \geq K_0, i = 1, 2, \dots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \dots, K-1$.

Moreover, since $t \in [n\Delta t, (n+1)\Delta t]$, by Lemma 2.8 we have

$$\left| \xi_{(i)j}^{(K)}(t) \right| \leq M_1 (\Delta t)^2 = \frac{M_1 T^2}{K^2} \quad \text{for } K \geq K_0, i = 1, 2, \dots, N, j = 1, 2 \text{ and } t \in [0, T].$$

Therefore, we obtain

$$\xi_{(i)j}^{(K)} \rightarrow 0 \text{ in } C([0, T]) \text{ as } K \rightarrow \infty \quad \text{for } i = 1, 2, \dots, N \text{ and } j = 1, 2.$$

We see that $\xi_{(i)}^{(K)} \in C([0, T])^2$ for $K \geq K_0, i = 1, 2, \dots, N$ and $\left| \xi_{(i)}^{(K)} \right| \leq 2M_1 (\Delta t)^2$ on $[0, T]$ for $K \geq K_0$ and $i = 1, 2, \dots, N$. This guarantees that Lemma 2.9 holds. \square

Proof of Theorem 2.3. From Theorem 2.1, we have the following integral equation

$$X_{(i)}(t) = \int_0^t V_{(i)}(\tau) d\tau + X_{0(i)} \quad \text{for } t \in [0, T] \text{ and } i = 1, 2, \dots, N.$$

By using this equation and (2.4.3), we have

$$X_{(i)}(t) - X_{(i)}^{(K)}(t) = \int_0^t \left(V_{(i)}(\tau) - V_{(i)}^{(K)}(\tau) \right) d\tau - \xi_{(i)}^{(K)}(t), \quad (2.4.6)$$

for $t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \dots, K-1, K \geq K_0$ and $i = 1, 2, \dots, N$.

Now, $V_{(i)}(0) = V_{(i)}^{(K)}(0)$ holds for $K \geq K_0$ and $i = 1, 2, \dots, N$, so we have

$$\begin{aligned}
&V_{(i)}(t) - V_{(i)}^{(K)}(t) \\
&= V_{(i)}(t) - V_{(i)}(0) + V_{(i)}^{(K)}(0) - V_{(i)}^{(K)}(t) \\
&= \int_0^{n\Delta t} \left(\frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) \right) d\tau + \int_{n\Delta t}^t \left(\frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) \right) d\tau \\
&= \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left(\frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) \right) d\tau + \int_{n\Delta t}^t \left(\frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) \right) d\tau, \quad (2.4.7)
\end{aligned}$$

for $i = 1, 2, \dots, N, K \geq K_0, n = 0, 1, \dots, K-1$ and $t \in [n\Delta t, (n+1)\Delta t]$.

By using (1.3.2) and (1.3.5), then we have

$$\begin{aligned}
\frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) &= \frac{1}{m} \left\{ f(\varepsilon_{(i)}) \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} - f(\varepsilon_{(i-1)}) \frac{X_{(i)}(\tau) - X_{(i-1)}(\tau)}{l_{(i-1)}(\tau)} \right\} \\
&\quad - \frac{\kappa}{4m} \left\{ \left\{ \varepsilon_{(i)}^{(p)} + \varepsilon_{(i)}^{(p-1)} + 1 - \frac{1}{(1 + \varepsilon_{(i)}^{(p)}) (1 + \varepsilon_{(i)}^{(p-1)})} \right\} \right. \\
&\quad \times \frac{X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) + X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau)}{\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right|} \left. \right\} \\
&\quad + \frac{\kappa}{4m} \left\{ \left\{ \varepsilon_{(i-1)}^{(p)} + \varepsilon_{(i-1)}^{(p-1)} + 1 - \frac{1}{(1 + \varepsilon_{(i-1)}^{(p)}) (1 + \varepsilon_{(i-1)}^{(p-1)})} \right\} \right. \\
&\quad \times \frac{X_{(i)}^{(p)}(\tau) - X_{(i-1)}^{(p)}(\tau) + X_{(i)}^{(p-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau)}{\left| X_{(i)}^{(p)}(\tau) - X_{(i-1)}^{(p)}(\tau) \right| + \left| X_{(i)}^{(p-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau) \right|} \left. \right\} \\
&\quad \text{for } i = 1, 2, \dots, N, K \geq K_0, \tau \in [(p-1)\Delta t, p\Delta t] \text{ and } p = 1, 2, \dots, n.
\end{aligned}$$

Accordingly, we have

$$\left| \frac{dV_{(i)}}{d\tau}(\tau) - \frac{dV_{(i)}^{(K)}}{d\tau}(\tau) \right| \leq F_1 + F_2,$$

for $i = 1, 2, \dots, N, K \geq K_0, \tau \in [(p-1)\Delta t, p\Delta t]$ and $p = 1, 2, \dots, n$, where

$$\begin{aligned}
F_1 &:= \left| \frac{1}{m} f(\varepsilon_{(i)}) \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} - \frac{\kappa}{4m} \left\{ \varepsilon_{(i)}^{(p)} + \varepsilon_{(i)}^{(p-1)} + 1 - \frac{1}{(1 + \varepsilon_{(i)}^{(p)}) (1 + \varepsilon_{(i)}^{(p-1)})} \right\} \right. \\
&\quad \times \frac{X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) + X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau)}{\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right|} \left. \right|, \\
F_2 &:= \left| \frac{1}{m} f(\varepsilon_{(i-1)}) \frac{X_{(i)}(\tau) - X_{(i-1)}(\tau)}{l_{(i-1)}(\tau)} - \frac{\kappa}{4m} \left\{ \varepsilon_{(i-1)}^{(p)} + \varepsilon_{(i-1)}^{(p-1)} + 1 - \frac{1}{(1 + \varepsilon_{(i-1)}^{(p)}) (1 + \varepsilon_{(i-1)}^{(p-1)})} \right\} \right. \\
&\quad \times \frac{X_{(i)}^{(p)}(\tau) - X_{(i-1)}^{(p)}(\tau) + X_{(i)}^{(p-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau)}{\left| X_{(i)}^{(p)}(\tau) - X_{(i-1)}^{(p)}(\tau) \right| + \left| X_{(i)}^{(p-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau) \right|} \left. \right|.
\end{aligned}$$

For F_1 we have

$$\begin{aligned}
F_1 &\leq \frac{\kappa}{4m} \left| \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} \left\{ \left(2\varepsilon_{(i)} + 1 - \frac{1}{(\varepsilon_{(i)} + 1)^2} \right) \right. \right. \\
&\quad \left. \left. - \left(\varepsilon_{(i)}^{(p)} + \varepsilon_{(i)}^{(p-1)} + 1 - \frac{1}{(1 + \varepsilon_{(i)}^{(p)}) (1 + \varepsilon_{(i)}^{(p-1)})} \right) \right\} \right|
\end{aligned}$$

$$+ \frac{\kappa}{4m} \left| \left\{ \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} - \frac{X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) + X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau)}{\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right|} \right\} \right. \\ \left. \times \left(\varepsilon_{(i)}^{(p)} + \varepsilon_{(i)}^{(p-1)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(p)}\right) \left(1 + \varepsilon_{(i)}^{(p-1)}\right)} \right) \right|$$

$$=: F_{1,1} + F_{1,2}.$$

For $F_{1,1}$ we have

$$F_{1,1} \\ \leq \frac{\kappa}{4m} \left\{ \left| \varepsilon_{(i)} - \varepsilon_{(i)}^{(p)} \right| + \left| \varepsilon_{(i)} - \varepsilon_{(i)}^{(p-1)} \right| + \left| \frac{\left(\varepsilon_{(i)} + 1 \right)^2 - \left(1 + \varepsilon_{(i)}^{(p)} \right) \left(1 + \varepsilon_{(i)}^{(p-1)} \right)}{\left(\varepsilon_{(i)} + 1 \right)^2 \left(1 + \varepsilon_{(i)}^{(p)} \right) \left(1 + \varepsilon_{(i)}^{(p-1)} \right)} \right| \right\} \\ = \frac{\kappa}{4m} \left\{ \left| \frac{l_{(i)}(\tau) - l_{(i)}^{(p)}(\tau)}{l_{N*}} \right| + \left| \frac{l_{(i)}(\tau) - l_{(i)}^{(p-1)}(\tau)}{l_{N*}} \right| \right. \\ \left. + \left| \frac{l_{N*}^2}{l_{(i)}^2(\tau) l_{(i)}^{(p)}(\tau) l_{(i)}^{(p-1)}(\tau)} \left\{ l_{(i)}^{(p)}(\tau) \left(l_{(i)}(\tau) - l_{(i)}^{(p-1)}(\tau) \right) + l_{(i)}(\tau) \left(l_{(i)}(\tau) - l_{(i)}^{(p)}(\tau) \right) \right\} \right| \right\} \\ \leq \frac{\kappa}{4m} \left\{ \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\left| l_{(i)}(\tau) \right| \left| l_{(i)}^{(p)}(\tau) \right| \left| l_{(i)}^{(p-1)}(\tau) \right|} \right) \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| \right. \\ \left. + \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\left| l_{(i)}(\tau) \right| \left| l_{(i)}^{(p)}(\tau) \right| \left| l_{(i)}^{(p-1)}(\tau) \right|} \right) \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right\} \\ + \frac{\kappa}{4m} \left\{ \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\left| l_{(i)}(\tau) \right|^2 \left| l_{(i)}^{(p-1)}(\tau) \right|} \right) \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| \right. \\ \left. + \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\left| l_{(i)}(\tau) \right|^2 \left| l_{(i)}^{(p-1)}(\tau) \right|} \right) \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right\}.$$

By using (2.3.5) and Lemmas 2.1 and 2.2, we have

$$F_{1,1} \leq \frac{\kappa}{4m} \left\{ \frac{1}{l_{N*}} + \frac{(4d_0 + N\kappa l_{N*})^3}{\kappa^3 l_{N*}^6} \right\} \left\{ \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\ \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right\},$$

where $d_0 = G(X_0, V_0)$. Next, for $F_{1,2}$ we have

$$F_{1,2} = \frac{\kappa}{4m} \left| \left\{ \frac{X_{(i+1)}(\tau) - X_{(i)}(\tau)}{l_{(i)}(\tau)} - \frac{X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) + X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau)}{\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right|} \right\} \right|$$

$$\begin{aligned}
& \times \left(\varepsilon_{(i)}^{(p)} + \varepsilon_{(i)}^{(p-1)} + 1 - \frac{1}{\left(1 + \varepsilon_{(i)}^{(p)}\right) \left(1 + \varepsilon_{(i)}^{(p-1)}\right)} \right) \Bigg| \\
& \leq \frac{\kappa}{4m} \left\{ \left| \varepsilon_{(i)}^{(p)} \right| + \left| \varepsilon_{(i)}^{(p-1)} \right| + 1 + \left| \frac{1}{\left(1 + \varepsilon_{(i)}^{(p)}\right) \left(1 + \varepsilon_{(i)}^{(p-1)}\right)} \right| \right\} \times \frac{\left| F_{1,2,1} \right| + \left| F_{1,2,2} \right|}{Q},
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{Q} &= \frac{1}{2 \left| X_{(i+1)}(\tau) - X_{(i)}(\tau) \right| \left(\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right)}, \\
F_{1,2,1} &= 2 \left(X_{(i+1)}(\tau) - X_{(i)}(\tau) \right) \\
&\quad \times \left(\left| X_{(i+1)}^{(p)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i+1)}^{(p-1)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| - 2 \left| X_{(i+1)}(\tau) - X_{(i)}(\tau) \right| \right), \\
F_{1,2,2} &= 2 \left| X_{(i+1)}(\tau) - X_{(i)}(\tau) \right| \\
&\quad \times \left(2X_{(i+1)}(\tau) - 2X_{(i)}(\tau) - X_{(i+1)}^{(p)}(\tau) + X_{(i)}^{(p)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) + X_{(i)}^{(p-1)}(\tau) \right).
\end{aligned}$$

Easily, we get

$$\begin{aligned}
\left| F_{1,2,1} \right| &\leq 2 \left| X_{(i+1)}(\tau) - X_{(i)}(\tau) \right| \left\{ \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\
&\quad \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right\}, \\
\left| F_{1,2,2} \right| &\leq 2 \left| X_{(i+1)}(\tau) - X_{(i)}(\tau) \right| \left(\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\
&\quad \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right).
\end{aligned}$$

By using (2.3.12), (2.3.6) in Lemma 2.6 and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
& F_{1,2} \\
& \leq \frac{\kappa}{4m} \left\{ \frac{\left| l_{(i)}^{(p)}(\tau) \right| + l_{N*}}{l_{N*}} + \frac{\left| l_{(i)}^{(p-1)}(\tau) \right| + l_{N*}}{l_{N*}} + 1 + \frac{l_{N*}^2}{\left| l_{(i)}^{(p)}(\tau) \right| \left| l_{(i)}^{(p-1)}(\tau) \right|} \right\} \\
& \quad \times \frac{(4d_0 + N\kappa l_{N*})^2}{\kappa^2 l_{N*}^4} \left(\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\
& \quad \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right) \\
& \leq \frac{4d_0 + N\kappa l_{N*}}{4m\kappa^2 l_{N*}^5} \left\{ 2\kappa l_{N*}^2 (4d_0 + N\kappa l_{N*}) + 3\kappa^2 l_{N*}^3 + (4d_0 + N\kappa l_{N*})^2 \right\} \\
& \quad \times \left(\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\
& \quad \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right).
\end{aligned}$$

Therefore, we have

$$F_1 \leq \beta \left(\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| \right. \\ \left. + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right) \quad \text{for } i = 1, 2, \dots, N.$$

where

$$\beta = \frac{1}{4m\kappa^2 l_{N*}^6} \left\{ \left(\kappa^3 l_{N*}^5 + (4d_0 + N\kappa l_{N*})^3 \right) \right. \\ \left. + l_{N*} (4d_0 + N\kappa l_{N*}) \left\{ 2\kappa l_{N*}^2 (4d_0 + N\kappa l_{N*}) + 3\kappa^2 l_{N*}^3 + (4d_0 + N\kappa l_{N*})^2 \right\} \right\}.$$

In the same way for F_1 , we have

$$F_2 \leq \beta \left(\left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p)}(\tau) \right| \right. \\ \left. + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| + \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau) \right| \right) \quad \text{for } i = 1, 2, \dots, N.$$

Therefore, we obtain

$$\left| \frac{dV_{(i)}(\tau)}{d\tau} - \frac{dV_{(i)}^{(K)}(\tau)}{d\tau} \right| \leq 2\beta \left(\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| + \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| \right. \\ \left. + \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| + \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| \right. \\ \left. + \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p)}(\tau) \right| + \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau) \right| \right), \quad (2.4.8)$$

for $K \geq K_0, i = 1, 2, \dots, N, \tau \in [(p-1)\Delta t, p\Delta t]$ and $p = 1, 2, \dots, K$.

By using (2.4.6), (2.4.7) and (2.4.8), we have

$$\left| X_{(i)}(t) - X_{(i)}^{(K)}(t) \right| \\ \leq \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| \frac{dV_{(i)}(\tau)}{d\tau} - \frac{dV_{(i)}^{(K)}(\tau)}{d\tau} \right| d\tau + \int_{n\Delta t}^t \left| \frac{dV_{(i)}(\tau)}{d\tau} - \frac{dV_{(i)}^{(K)}(\tau)}{d\tau} \right| d\tau \right) ds + \left| \xi_{(i)}^{(K)}(t) \right| \\ \leq 2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(n+1)}(\tau) \right| d\tau \right) ds \\ + 2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(n)}(\tau) \right| d\tau \right) ds \\ + 2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i)}(\tau) - X_{(i)}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i)}(\tau) - X_{(i)}^{(n+1)}(\tau) \right| d\tau \right) ds \\ + 2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i)}(\tau) - X_{(i)}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i)}(\tau) - X_{(i)}^{(n)}(\tau) \right| d\tau \right) ds$$

$$\begin{aligned}
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(n+1)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(n)}(\tau) \right| d\tau \right) ds \\
& + \left| \xi_{(i)}^{(K)}(t) \right| \\
& =: R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + \left| \xi_{(i)}^{(K)}(t) \right|, \tag{2.4.9}
\end{aligned}$$

for $K \geq K_0, i = 1, 2, \dots, N, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \dots, K-1$.

For R_1 we have

$$\begin{aligned}
\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| & \leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + \left| X_{(i+1)}^{(K)}(\tau) - X_{(i+1)}^{(K)}(p\Delta t) \right| \\
& \leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + \int_{\tau}^{p\Delta t} \left| \frac{dX_{(i+1)}^{(K)}}{du}(u) \right| du,
\end{aligned}$$

for $K \geq K_0, i = 1, 2, \dots, N, p = 0, 1, \dots, K-1$ and $\tau \in [(p-1)\Delta t, p\Delta t]$.

By using (2.4.2), (1.3.6) and Lemma 2.8, we have

$$\left| \frac{dX_{(i+1)}^{(K)}}{du}(u) \right| \leq M_1 \quad \text{for } u \in [\tau, p\Delta t], K \geq K_0 \text{ and } i = 1, 2, \dots, N. \tag{2.4.10}$$

Therefore, we have

$$\begin{aligned}
\left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p)}(\tau) \right| & \leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + M_1 |p\Delta t - \tau|, \\
& \text{for } K \geq K_0, i = 1, 2, \dots, N, p = 0, 1, \dots, K-1 \text{ and } \tau \in [(p-1)\Delta t, p\Delta t].
\end{aligned}$$

By using these inequalities, we have

$$\begin{aligned}
R_1 & \leq 2\beta \int_0^t \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} |p\Delta t - \tau| d\tau ds \\
& \quad + 2\beta \int_0^t \int_{n\Delta t}^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \int_{n\Delta t}^t ((n+1)\Delta t - \tau) d\tau ds \\
& \leq 2\beta \int_0^t \int_0^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + (2\beta M_1 T^2) \Delta t \\
& \quad + 2\beta M_1 T \left((n+1)\Delta t(t - n\Delta t) - \frac{1}{2}(t - n\Delta t)(t + n\Delta t) \right) \\
& \leq 2\beta T \int_0^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + 2\beta M_1 T (\Delta t)^2.
\end{aligned}$$

In the same way for R_1 , we have

$$\begin{aligned} R_3 &\leq 2\beta T \int_0^t \left| X_{(i)}(\tau) - X_{(i)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + 2\beta M_1 T (\Delta t)^2, \\ R_5 &\leq 2\beta T \int_0^t \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + 2\beta M_1 T (\Delta t)^2. \end{aligned}$$

Next, for R_2 by using (2.4.10), we have

$$\begin{aligned} \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(p-1)}(\tau) \right| &\leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + \left| X_{(i+1)}^{(K)}(\tau) - X_{(i+1)}^{(K)}((p-1)\Delta t) \right| \\ &\leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + \int_{(p-1)\Delta t}^{\tau} \left| \frac{dX_{(i+1)}^{(K)}}{du}(u) \right| du \\ &\leq \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| + M_1 |\tau - (p-1)\Delta t|, \end{aligned}$$

for $K \geq K_0, i = 1, 2, \dots, N, p = 0, 1, \dots, K-1$ and $\tau \in [(p-1)\Delta t, p\Delta t]$.

By using these inequalities, we have

$$\begin{aligned} R_2 &\leq 2\beta \int_0^t \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} |\tau - (p-1)\Delta t| d\tau ds \\ &\quad + 2\beta \int_0^t \int_{n\Delta t}^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \int_{n\Delta t}^t (\tau - n\Delta t) d\tau ds \\ &\leq 2\beta \int_0^t \int_0^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau ds + (2\beta M_1 T^2) \Delta t \\ &\quad + 2\beta M_1 T \left(\frac{1}{2} (t + n\Delta t) (t - n\Delta t) - n\Delta t (t - n\Delta t) \right) \\ &\leq 2\beta T \int_0^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2. \end{aligned}$$

In the same way for R_2 , we have

$$\begin{aligned} R_4 &\leq 2\beta T \int_0^t \left| X_{(i)}(\tau) - X_{(i)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2, \\ R_6 &\leq 2\beta T \int_0^t \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(K)}(\tau) \right| d\tau + (2\beta M_1 T^2) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2. \end{aligned}$$

Therefore, from the estimates for R_i for $i = 1, 2, 3, 4, 5, 6$, Lemma 2.9 and (2.4.9) it follows

$$\begin{aligned}
& \left| X_{(i)}(t) - X_{(i)}^{(K)}(t) \right| \\
& \leq 4\beta T \left\{ \int_0^t \left| X_{(i+1)}(\tau) - X_{(i+1)}^{(K)}(\tau) \right| d\tau + \int_0^t \left| X_{(i)}(\tau) - X_{(i)}^{(K)}(\tau) \right| d\tau + \int_0^t \left| X_{(i-1)}(\tau) - X_{(i-1)}^{(K)}(\tau) \right| d\tau \right\} \\
& \quad + 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2, \tag{2.4.11} \\
& \quad \text{for } K \geq K_0, i = 1, 2, \dots, N \text{ and } t \in [0, T].
\end{aligned}$$

By taking the sum of both side of (2.4.11) from $i = 1$ to N , we obtain

$$\begin{aligned}
\sum_{i=1}^N \left| X_{(i)}(t) - X_{(i)}^{(K)}(t) \right| & \leq 12\beta T \int_0^t \sum_{i=1}^N \left| X_{(i)}(\tau) - X_{(i)}^{(K)}(\tau) \right| d\tau \\
& \quad + N \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\}, \\
& \quad \text{for } K \geq K_0 \text{ and } t \in [0, T].
\end{aligned}$$

Put $E_K(t) = \sum_{i=1}^N \left| X_{(i)}(t) - X_{(i)}^{(K)}(t) \right|$ for $K \geq K_0$ and $t \in [0, T]$. Easily, we get

$$\begin{aligned}
E_K(t) & \leq 12\beta T \int_0^t E_K(\tau) d\tau + N \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\}, \\
& \quad \text{for } K \geq K_0 \text{ and } t \in [0, T].
\end{aligned}$$

By using Gronwall's inequality, we obtain

$$\begin{aligned}
E_K(t) & \leq N \left(12\beta T^2 e^{12\beta T} + 1 \right) \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\} \tag{2.4.12} \\
& \rightarrow 0 \quad \text{as } K \rightarrow \infty \quad \text{for } t \in [0, T].
\end{aligned}$$

We see that $E_K \rightarrow 0$ in $C([0, T])$ as $K \rightarrow \infty$, namely,

$$X_{(i)}^{(K)} \rightarrow X_{(i)} \text{ in } C([0, T])^2 \text{ as } K \rightarrow \infty \quad \text{for } i = 1, 2, \dots, N.$$

Moreover, by using (2.4.12), we see that

$$\begin{aligned}
\left| X_{(i)}(t) - X_{(i)}^{(K)}(t) \right| & \leq N \left(12\beta T^2 e^{12\beta T} + 1 \right) \left\{ 12\beta M_1 T^2 + 9\beta M_1 T (\Delta t) + 2M_1 (\Delta t) \right\} \Delta t, \\
& \quad \text{for } K \geq K_0, i = 1, 2, \dots, N \text{ and } t \in [0, T].
\end{aligned}$$

Thus, Theorem 2.3 has been proved. \square

2.5 Periodic solutions

By using the following Lemmas 2.10–2.12, we can prove Proposition 2.1. For simplicity, we put

$$a = \frac{\sqrt{\alpha\beta\kappa}}{2}, b = \frac{1}{2\alpha}, c = \sqrt{\frac{m}{2}}, k = \frac{\beta\kappa}{4\alpha^2}, l = \frac{\beta\kappa}{16\alpha} + \frac{\beta\kappa}{4} \left(\alpha R_0^2 - R_0 + \frac{1}{\alpha^2 R_0} \right) + \frac{m}{2} |v_0|^2 \text{ and } v_0 \in \mathbb{R}.$$

Here, we consider the following equation:

$$a^2(x-b)^2 + \frac{k}{x} + c^2 y^2 = l \quad \text{for } x > 0 \text{ and } y \in \mathbb{R}. \quad (2.5.1)$$

Lemma 2.10. *If $R_0 > 0$, then there exist positive constants α_1 and α_2 with $\alpha_1 < \alpha_2$ satisfying*

(i) *For all $x \in (\alpha_1, \alpha_2)$ there exists $y > 0$ such that (2.5.1) holds.*

(ii) *$(\alpha_1, 0)$ and $(\alpha_2, 0)$ satisfy (2.5.1).*

Proof. Put $\varphi(x) = a^2(x-b)^2 + \frac{k}{x}$ for all $x > 0$. We shall show existence of an interval $[\alpha_1, \alpha_2]$ of x such that $\varphi(x) \leq l$ hold for $x \in [\alpha_1, \alpha_2]$. We have

$$\begin{aligned} \varphi'(x) &= \frac{\beta\kappa}{4\alpha^2 x^2} (2\alpha^3 x^3 - \alpha^2 x^2 - 1) \\ &= \frac{\beta\kappa}{4\alpha^2 x^2} \left(x - \frac{1}{\alpha} \right) (2\alpha^3 x^2 + \alpha^2 x + \alpha) \quad \text{for } x > 0, \end{aligned}$$

and

$$\varphi'(x) \begin{cases} < 0 & \text{if } 0 < x < \frac{1}{\alpha}, \\ = 0 & \text{if } x = \frac{1}{\alpha}, \\ > 0 & \text{if } x > \frac{1}{\alpha}. \end{cases} \quad (2.5.2)$$

Also, we have

$$\lim_{x \downarrow 0} \varphi(x) = +\infty, \quad \lim_{x \rightarrow +\infty} \varphi(x) = +\infty,$$

and

$$\varphi\left(\frac{1}{\alpha}\right) = \frac{5\beta\kappa}{16\alpha}, \quad l - \varphi\left(\frac{1}{\alpha}\right) = \frac{43\beta\kappa}{16\alpha} + \frac{\beta\kappa}{4\alpha^2 R_0} + \frac{m}{2} |v_0|^2 > 0.$$

Therefore, since φ is continuous on $(0, \infty)$ we see that there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 < \alpha < \alpha_2 \quad \text{and} \quad \varphi(\alpha_1) = \varphi(\alpha_2) = l.$$

Furthermore, by using (2.5.2), $\varphi(x) < l$ holds for $\alpha_1 < x < \alpha_2$. Thus, this lemma has been proved. \square

Lemma 2.11. $P(R_0, v_0)$ has a unique solution $R \in C^2([0, \infty))$ and there exists a positive constant M such that $\frac{1}{M} \leq R(t) \leq M$ for all $t \in [0, \infty)$.

Proof. Since g is locally Lipschitz continuous on $(0, \infty)$, there exist $T_1 > 0$ and a unique solution $R_1 \in C^2([0, T_1])$ of $P(R_0, v_0)$ on $[0, T_1]$, that is,

$$\begin{aligned} mR_1'' &= -\beta g(R_1) \text{ on } [0, T_1], \\ R_1(0) &= R_0, \quad R_1'(0) = v_0. \end{aligned} \tag{2.5.3}$$

By multiplying (2.5.3) with R_1' , we see that the following energy is preserved:

$$\frac{d}{dt} \left(\frac{m}{2} |R_1'|^2 + \beta G(R_1) \right) = 0 \text{ on } [0, T_1], \tag{2.5.4}$$

where $G(R_1) = \frac{\kappa}{2} \left(\frac{\alpha}{2} R_1^2 - \frac{R_1}{2} + \frac{1}{2\alpha^2 R_1} \right)$.

Here, we can show $G(R) > 0$ for $R > 0$. Indeed, since it holds

$$G(R) = \frac{\kappa}{2} \left\{ \frac{\alpha}{2} \left(R - \frac{1}{2\alpha} \right)^2 + \frac{1}{2\alpha} \left(\frac{1}{\alpha R} - \frac{1}{4} \right) \right\} \quad \text{for } R > 0,$$

we see that $G(R) > 0$ holds for all $0 < R < \frac{4}{\alpha}$. If $R \geq \frac{4}{\alpha}$, then

$$\left(R - \frac{1}{2\alpha} \right)^2 \geq \left(\frac{4}{\alpha} - \frac{1}{2\alpha} \right)^2, \text{ and } G(R) > 0.$$

Accordingly, by putting $d_0 = \frac{m}{2} |v_0|^2 + \beta g(G(R_0))$, we have $d_0 > 0$. From (2.5.4) and $d_0 > 0$, it follows that

$$R_1(t) \geq \frac{1}{2\alpha^2 \left(\frac{2d_0}{\beta\kappa} + \frac{1}{8\alpha} \right)} > 0 \quad \text{for all } t \in [0, T_1]. \tag{2.5.5}$$

Since (R_1, R_1') satisfies (2.5.1), Lemma 2.10 implies that (R_1, R_1') is on the bounded closed curve which depends only on the initial data and the given values. Clearly, by combining (2.5.5), there exists a positive constant M independent of T_1 such that $\frac{1}{M} \leq R_1(t) \leq M$ for $t \in [0, T_1]$. Hence, we can extend the solution R_1 beyond T_1 . Therefore, we obtain a unique solution $R \in C^2([0, \infty))$ of $P(R_0, v_0)$ on $[0, \infty)$. Thus, this lemma is true. \square

We can easily show the following lemma. So, we omit its proof.

Lemma 2.12. $g\left(\frac{1}{\alpha}\right) = 0$, $g(R) > 0$ for $0 < R < \frac{1}{\alpha}$, and $g(R) > 0$ for $R > \frac{1}{\alpha}$ holds. Moreover, $g'(R) > 0$ for $R > 0$.

Here, we give a proof of Proposition 2.1 by applying Lemmas 2.10, 2.11 and 2.12.

Proof of Proposition 2.1. Put $v = R'$ on $[0, \infty)$. For the sake of simplicity, we assume that $v_0 = 0$ and $0 < R_0 < \frac{1}{2\alpha}$. By Lemma 2.11, $P(R_0, v_0)$ has the unique solution $R \in C^2([0, \infty))$ on $[0, \infty)$, then we obtain

$$v'(0) = \frac{\beta\kappa}{2m} \left\{ \frac{1}{2\alpha^2 R_0^2} + \alpha \left(\frac{1}{2\alpha} - R_0 \right) \right\} > 0.$$

Now, since v' is continuous at $t = 0$, there exist $\delta_1 > 0$ and $T_1 > 0$ such that

$$v' \geq \delta_1 \text{ on } [0, T_1] \text{ and } v(t) \geq \delta_1 t \text{ for } t \in [0, T_1], \text{ and } R(t) \geq R_0 + \frac{\delta_1}{2} t \text{ for } t \in [0, T_1].$$

Here, we suppose that $v' > 0$ on $[T_1, \infty)$. It is easy to see that

$$v(t) \geq v(T_1) \geq \delta_1 T_1 > 0 \quad \text{for all } t \in [T_1, \infty).$$

Accordingly, we have

$$R(t) \geq R(T_1) + \delta_1 T_1 (t - T_1) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

This is a contradiction for the boundedness of R shown in Lemma 2.11. Therefore, there exists $\widehat{T}_2 > T_1$ such that $v'(\widehat{T}_2) = 0$. This shows that the set $E_1 = \{t > T_1 \mid v'(t) = 0\}$ is not empty. Thus, we can put $T_2 = \inf E_1$. Since v' is continuous on $[0, \infty)$, it holds that

$$v'(T_2) = 0 \text{ and } v' > 0 \text{ on } [T_1, T_2), \quad (2.5.6)$$

and

$$v(T_2) > v(T_1) > \delta_1 T_1 > 0. \quad (2.5.7)$$

By (2.5.6), we have $g(R(T_2)) = 0$. On account of Lemma 2.12, we get $R(T_2) = \frac{1}{\alpha}$. The continuity of v at $t = T_2$ and (2.5.7) guarantee existence of $\delta_2 > 0$ and $b > 0$ such that

$$v \geq b \quad \text{on } [T_2, T_2 + \delta_2] \quad \text{and} \quad R(T_2 + b\delta_2) > \frac{1}{\alpha}.$$

Therefore, by applying Lemma 2.12 again, we have $g(R(T_2 + \delta_2)) > 0$.

Here, we assume that $v > 0$ on $[T_2 + \delta_2, \infty)$. Easily, we get

$$R(t) \geq R(T_2 + \delta_2) > \frac{1}{\alpha} \quad \text{for } t \in [T_2 + \delta_2, \infty).$$

Moreover, by using Lemma 2.12 again, we obtain

$$mv'(t) = -\beta g(R(t)) \leq -\beta g(R(T_2 + \delta_2)) < 0 \text{ for } t \in [T_2 + \delta_2, \infty),$$

and there exists a positive constant q such that $v'(t) \leq -q$ for all $t \in [T_2 + \delta_2, \infty)$ and $v(t) \leq v(T_2 + \delta_2) - q\{t - (T_2 + \delta_2)\}$ for all $t \in [T_2 + \delta_2, \infty)$. Hence, the convergence

$$v(T_2 + \delta_2) - q\{t - (T_2 + \delta_2)\} \rightarrow -\infty \quad \text{as } t \rightarrow \infty,$$

leads to a contradiction. Accordingly, there exists $\hat{T}_3 > T_2 + \delta_2$ such that $v(\hat{T}_3) = 0$. Namely, the set $E_2 = \{t > T_2 + \delta_2 | v(t) = 0\}$ is not empty. Thus, we can put $T_3 = \inf E_2$. As mentioned above, $v(T_3) = 0$ and $v > 0$ on $[T_2 + \delta_2, T_3)$, and $R(T_3) > R(T_2 + \delta_2) > \frac{1}{\alpha}$. This means that (R, v) arrives at $(\alpha_2, 0)$ at $t = T_3$.

Similarly, we can show that (R, v) reaches $(R_0, 0)$ in a finite time. Hence, there exists $T_* > 0$ such that $R(T_*) = R_0$ and $v(T_*) = 0$.

The uniqueness of solutions to $P(R_0, 0)$ implies that R is periodic in time. Thus, this proposition has been proved. \square

2.6 Numerical results (1)

In this section, we shall show numerical results for (OP) by applying (NS) (see (1.3.5), (1.3.6))) and Euler's method (EM) with

$$N = 12, \Delta t = 0.001, \kappa = 0.1, m = \frac{5}{6}, l_* = 1,$$

$$X_{0(i)} = R_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, R_0 = 0.6 \text{ and } V_{0(i)} = v_0 A^i \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } i = 1, 2, \dots, N.$$

Figure 2.4 and Figure 2.6 indicate $R_i = |X_{(i)}|$ for $i = 1, 2, \dots, N$, by (EM) and (NS), respectively. Also, Figures 2.5 and 2.7 represent time variation of the energy $F(X, V)$ (see Theorem 2.2 and Lemma 2.5) calculated by (EM) and (NS), respectively.

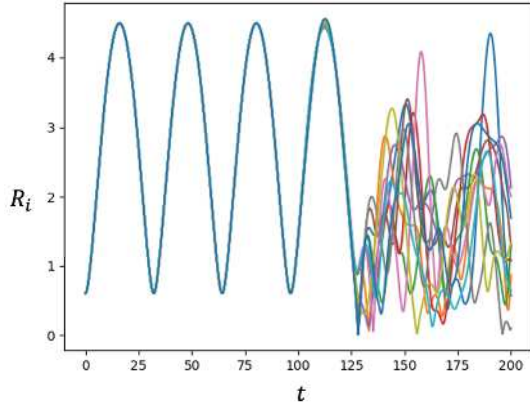


Figure 2.4: Time variation of radius for $N = 12$ and $\Delta t = 0.001$ by Euler's method

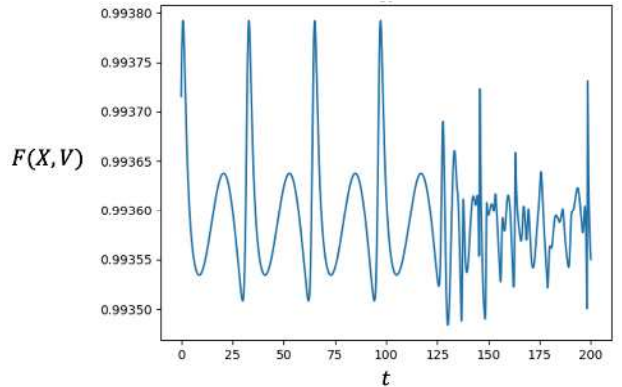


Figure 2.5: Time variation of the energy for $N = 12$ and $\Delta t = 0.001$ by Euler's method

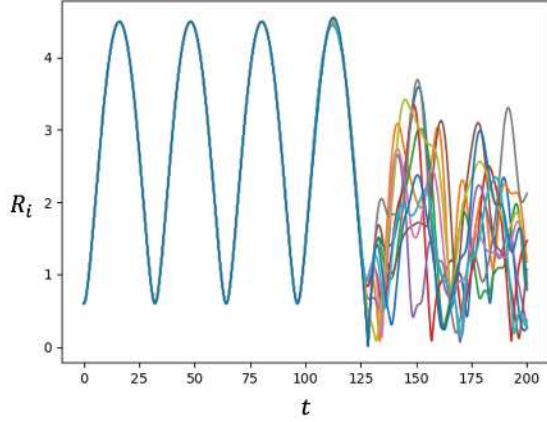


Figure 2.6: Time variation of radius for $N = 12$ and $\Delta t = 0.001$ by using (NS)

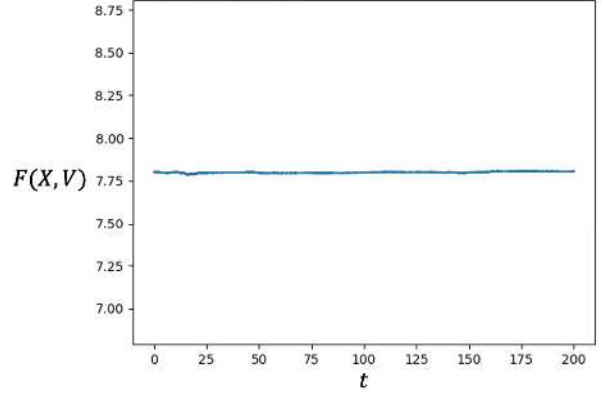


Figure 2.7: Time variation of the energy for $N = 12$ by using (NS), where the value of the vertical axis indicates $F(X, V) = (\text{the value}) \times 10^{-12} + 0.99371558636$

From observation to these numerical results we infer the following remarks.

- The scheme (NS) conserves the energy F much better than (EM).
- By Proposition 2.1 the solution of (OP) must be periodic in time for spherically symmetric initial conditions. Up to time $t = 100$, the numerical solutions seem to be periodic in both cases. However, after passing $t = 125$, the solutions do not keep periodicity, in these simulations.
- From the behaviors of R_i , for $i = 1, 2, \dots, N$, it is difficult to find differences in the accuracy of the two methods, (NS) and (EM).

2.7 Numerical results (2)

As mentioned in the introduction, here we give a multi-steps structure-preserving scheme (MS) with the staggered time mesh $(n + 1/2)\Delta t$ ($n = 0, 1, \dots, K$) for discretizing the variable V .

(MS): Find $X^{(n+1)}$ and $V^{(n+3/2)}$ such that

$$\frac{X_{(i)}^{(n+1)} - X_{(i)}^{(n)}}{\Delta t} = V_{(i)}^{(n+\frac{1}{2})}, \quad (2.7.1)$$

$$\begin{aligned} & \frac{V_{(i)}^{(n+\frac{3}{2})} - V_{(i)}^{(n-\frac{1}{2})}}{2\Delta t} \\ &= -\frac{\kappa}{4m} \left\{ \varepsilon_{(i-1)}^{(n+1)} + \varepsilon_{(i-1)}^{(n)} + 1 - \frac{1}{(1 - \varepsilon_{(i-1)}^{(n+1)})(1 - \varepsilon_{(i-1)}^{(n)})} \right\} \frac{X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)} + X_{(i)}^{(n)} - X_{(i-1)}^{(n)}}{|X_{(i)}^{(n+1)} - X_{(i-1)}^{(n+1)}| + |X_{(i)}^{(n)} - X_{(i-1)}^{(n)}|} \\ &+ \frac{\kappa}{4m} \left\{ \varepsilon_{(i)}^{(n+1)} + \varepsilon_{(i)}^{(n)} + 1 - \frac{1}{(1 - \varepsilon_{(i)}^{(n+1)})(1 - \varepsilon_{(i)}^{(n)})} \right\} \frac{X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)} + X_{(i+1)}^{(n)} - X_{(i)}^{(n)}}{|X_{(i+1)}^{(n+1)} - X_{(i)}^{(n+1)}| + |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|} \end{aligned} \quad (2.7.2)$$

for all $n = 0, 1, \dots, K-1$ and $i = 1, 2, \dots, N$, where $X^{(n)}, V^{(n+1/2)}, V^{(n-1/2)}, \varepsilon_{(i)}^{(n)} = l_{(i)}^{(n)} - l_{N*}/l_{N*}$, and $l_{(i)}^{(n)} = |X_{(i+1)}^{(n)} - X_{(i)}^{(n)}|$ are given.

Proposition 2.2. *The solution of the scheme (2.7.1)–(2.7.2) satisfies*

$$F(X^{(n)}, V^{(n+\frac{1}{2})}, V^{(n-\frac{1}{2})}) = F(X^{(0)}, V^{(\frac{1}{2})}, V^{(-\frac{1}{2})}) \quad (n = 0, 1, \dots, K).$$

where

$$F(X^{(n)}, V^{(n+\frac{1}{2})}, V^{(n-\frac{1}{2})}) := \sum_{i=1}^N \left\{ \frac{m}{2} V_{(i)}^{(n+\frac{1}{2})} V_{(i)}^{(n-\frac{1}{2})} + l_{N*} \hat{f}(\varepsilon_{(i)}^{(n)}) \right\} \quad (n = 0, 1, \dots, K).$$

We can prove this proposition, easily. We omit its proof.

Remark 2.1. The above scheme is “explicit” if we compute $X^{(1)}, V^{(3/2)}, X^{(2)}, V^{(5/2)}, \dots$ in this order. By defining the above multi-step discrete energy and taking its variation based on DVDMM, we have designed the explicit structure-preserving scheme for (OP).

2.7.1 Numerical results

In this section, we show computation examples by our multi-steps structure-preserving scheme (MS) and demonstrate that the scheme inherits the conservative property from (OP) in a discrete sense. Also, we compare (MS) with (NS).

As initial data, we consider

$$\begin{aligned} X_{(i)}^{(0)} &= X_{0(i)} = R_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (i = 1, 2, \dots, N), \quad R_0 = 0.6, \\ V_{(i)}^{(\frac{1}{2})} &= V_{(i)}^{(0)} - \frac{\kappa \Delta t}{8m} \left\{ 2\varepsilon_{(i-1)}^{(0)} + 1 - \frac{1}{(1 - \varepsilon_{(i-1)}^{(0)})^2} \right\} \frac{X_{(i)}^{(0)} - X_{(i-1)}^{(0)}}{|X_{(i)}^{(0)} - X_{(i-1)}^{(0)}|} \\ &\quad + \frac{\kappa \Delta t}{8m} \left\{ 2\varepsilon_{(i)}^{(0)} + 1 - \frac{1}{(1 - \varepsilon_{(i)}^{(0)})^2} \right\} \frac{X_{(i+1)}^{(0)} - X_{(i)}^{(0)}}{|X_{(i+1)}^{(0)} - X_{(i)}^{(0)}|} \quad (i = 1, 2, \dots, N), \end{aligned}$$

$$V_{(i)}^{(0)} = V_{0(i)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (i = 1, 2, \dots, N), \text{ and } V_{(i)}^{(-1/2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (i = 1, 2, \dots, N).$$

Also, we choose $N = 12$, $\Delta t = 0.001$, $\kappa = 0.1$, $m = 5/6$, and $l_* = 4\pi$. We note that the computer language used in this section is Julia (Version 1.5.3). Figures 2.8 and 2.9 show the time variation of the radius $R_i = |X_{(i)}|$ for $i = 1, 2, \dots, N$, by (NS) and (MS), respectively.

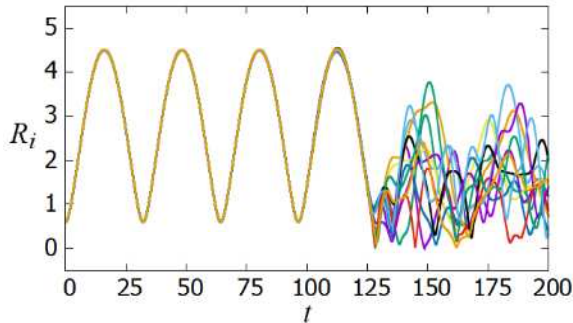


Figure 2.8: Time variation of radius for $N = 12$ by using (NS)

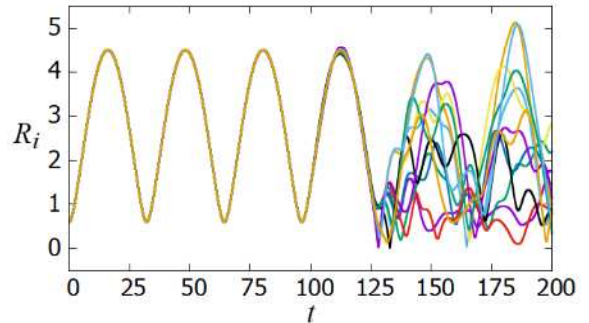


Figure 2.9: Time variation of radius for $N = 12$ by using (MS)

These figures show that the numerical solution obtained by (MS) is similar to the one by (NS) up to $t = 125$.

We note that the computation time by (NS) is 5463 seconds, i.e., about one and a half hours. On the other hand, the computation time by (MS) is 5 seconds. That is, the computation time by (MS) is much faster than that by (NS).

Next, we confirm the conservative property. Figure 2.10 shows the time variation of $M_{1d}^{(n)} := F(X^{(n)}, V^{(n)}) - F(X^{(0)}, V^{(0)})$ ($n = 0, 1, \dots, K$) calculated by (NS). Moreover, Figure 2.11 represents the time variation of $M_{2d}^{(n)} := F(X^{(n)}, V^{(n+1/2)}, V^{(n-1/2)}) - F(X^{(0)}, V^{(1/2)}, V^{(-1/2)})$ ($n = 0, 1, \dots, K$) calculated by (MS). From Theorems 2.2 and 2.3, we have $M_{1d}^{(n)} = 0$ and $M_{2d}^{(n)} = 0$ for all $n = 0, 1, \dots, K$.

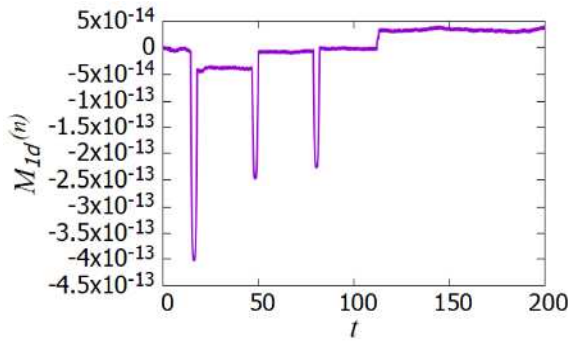


Figure 2.10: Time variation of $M_{1d}^{(n)}$ for $N = 12$ by using (NS): $M_{1d}^{(n)}$ does not change by about 12 orders of magnitude

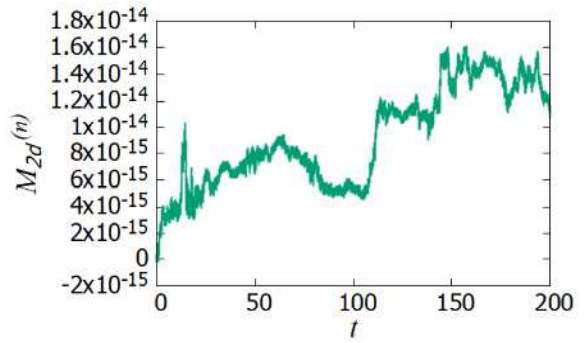


Figure 2.11: Time variation of $M_{2d}^{(n)}$ for $N = 12$ by using (MS): $M_{2d}^{(n)}$ does not change by about 13 orders of magnitude

These graphs show that the quantities $M_{1d}^{(n)}$ and $M_{2d}^{(n)}$ are conserved numerically. Besides, the scheme (MS) conserves the energy F much better than (NS).

Finally, we compare the relative errors up to $t = 100$. We denote the radii obtained by (NS) and (MS) by $R_{NS,i}$ and $R_{MS,i}$ for $i = 1, 2, \dots, N$, respectively. Also, we denote the numerical solution to the problem $P(R_0, 0)$ by Heun's method by $R_{H,i}$ ($i = 1, 2, \dots, N$). Figures 2.12 and 2.13 indicate the relative errors $|R_{NS,10} - R_{H,10}|/|R_{H,10}|$ and $|R_{MS,10} - R_{H,10}|/|R_{H,10}|$, respectively.

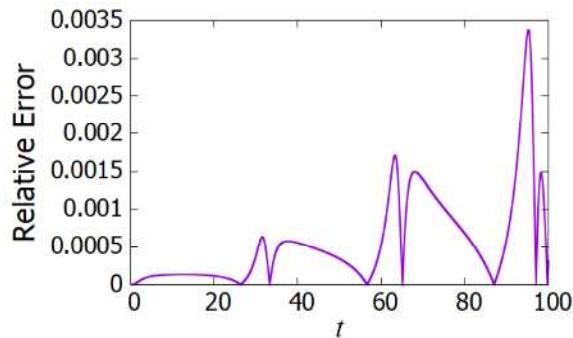


Figure 2.12: Time variation of relative error $|R_{NS,10} - R_{H,10}|/|R_{H,10}|$ by (NS)

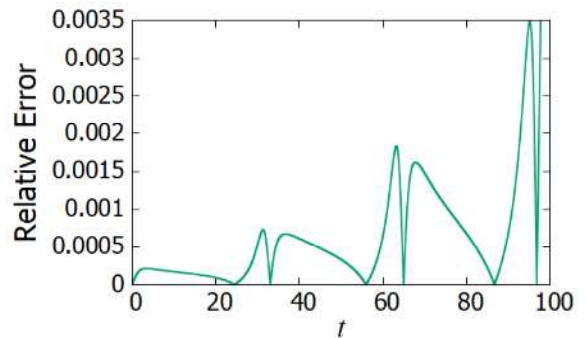


Figure 2.13: Time variation of relative error $|R_{MS,10} - R_{H,10}|/|R_{H,10}|$ by (MS)

From these figures, we can see that the relative error by (MS) is slightly larger than that by (NS) around $t = 100$, but there is no significant difference other than that. Even in the cases of $i = 0$ and $i = 5$, the relative error by (MS) is slightly larger than that by (NS) around $t = 100$. In the other cases, there is no major difference in the relative errors.

Chapter 3

The partial differential equation model with Lipschitz continuous stress function

3.1 Derivation of the partial differential equation model

In this section, we derive the partial differential equation model from (OP) (see Section 1.3) and show the numerical results for the model.

Let $M > 0$ be the total mass of the 1-dimensional elastic material occupied in the interval $[0, 1]$ as a natural state. Namely, the natural length of the material l_* is given by 1. From now on, we consider the motion of the closed ring made by the material (see Figure 1.7 in Chapter 1). We denote by $u(t, x) \in \mathbb{R}^2$ the position of the point which is at $x \in [0, 1]$ in the natural state for $t \in [0, T]$, $T > 0$.

The aim of this section is to derive the following hyperbolic equation from (3.1.2) in formal calculations:

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\frac{\partial u}{\partial x}}{\left| \frac{\partial u}{\partial x} \right|} \right) = 0, \quad \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1, \quad \text{in } Q(T), \quad (3.1.1)$$

where $\rho = \frac{M}{l_*}$ and $Q(T) = (0, T) \times (0, 1)$.

Let $x \in (0, 1)$, $t \in (0, T)$ and $N \in \mathbb{Z}_{>0}$. First, we suppose that the kinetic motion of the small part $(x - l_{N*}, x + l_{N*})$ is approximated by the similar ODE to (3.1.2) as follows:

$$\begin{aligned} m_N \frac{\partial^2 u(t, x)}{\partial t^2} &= f(\varepsilon_N(t, x)) \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} \\ &\quad - f(\varepsilon_N(t, x - l_{N*})) \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} + o\left(\frac{1}{N}\right) \text{ for } t \in [0, T], \end{aligned} \quad (3.1.2)$$

where

$$m_N = \frac{M}{N}, \quad l_N(t, x) = |u(t, x + l_{N*}) - u(t, x)|,$$

$$\varepsilon_N(t, x) = \frac{l_N(t, x) - l_{N*}}{l_{N*}} \text{ and } \lim_{N \rightarrow \infty} o\left(\frac{1}{N}\right) N = 0. \quad (3.1.3)$$

Moreover, in the argument below, we suppose that

- u is sufficiently smooth on $\overline{Q(T)}$.
- $u_x \neq 0$ on $\overline{Q(T)}$, $l_N > 0$ on $\overline{Q(T)}$ for each $N \in \mathbb{Z}_{>0}$.
- $f \in C^2((-1, \infty))$ and $\lim_{\varepsilon \downarrow -1} f(\varepsilon) = -\infty$.

The derivation for (3.1.1) is rather long. So, we divide it into several steps in the following. We fix $(t, x) \in Q(T)$ and sometimes omit it.

1st step. $\varepsilon_N \rightarrow \varepsilon$ as $N \rightarrow \infty$.

Proof of 1st step. Put $u = (u_1, u_2)$. Easily, we have

$$\begin{aligned} \varepsilon_N(t, x) &= \sqrt{\left(\frac{u_1(t, x + l_{N*}) - u_1(t, x)}{l_{N*}}\right)^2 + \left(\frac{u_2(t, x + l_{N*}) - u_2(t, x)}{l_{N*}}\right)^2} - 1 \\ &\rightarrow \sqrt{\left(\frac{\partial u_1(t, x)}{\partial x}\right)^2 + \left(\frac{\partial u_2(t, x)}{\partial x}\right)^2} - 1 = \left|\frac{\partial u(t, x)}{\partial x}\right| - 1 \\ &= \varepsilon(t, x) \quad \text{as } N \rightarrow \infty. \quad \square \end{aligned}$$

By dividing (3.1.2) by l_{N*} , we have

$$\begin{aligned} \rho \frac{\partial^2 u(t, x)}{\partial t^2} &= \frac{1}{l_{N*}} \left\{ f(\varepsilon_N(t, x)) \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} \right. \\ &\quad \left. - f(\varepsilon_N(t, x - l_{N*})) \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} \right\} + o\left(\frac{1}{N}\right) \frac{1}{l_{N*}}, \end{aligned}$$

and for each $N \in \mathbb{Z}_{>0}$, we put

$$\begin{aligned} I_N &= \frac{1}{l_{N*}} \left\{ f(\varepsilon_N(t, x)) \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} \right. \\ &\quad \left. - f(\varepsilon_N(t, x - l_{N*})) \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} \right\} + o\left(\frac{1}{N}\right) \frac{N}{l_*}. \end{aligned}$$

Clearly, it is sufficient to show the following convergence:

$$I_N \rightarrow \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\frac{\partial u}{\partial x}}{\left|\frac{\partial u}{\partial x}\right|} \right) \quad \text{as } N \rightarrow \infty. \quad (3.1.4)$$

For proving (3.1.4) we see that

$$\begin{aligned}
& \left| I_N - \frac{\partial}{\partial x} \left(f(\varepsilon(t, x)) \frac{\frac{\partial u}{\partial x}}{\left| \frac{\partial u}{\partial x} \right|} \right) \right| \\
& \leq \left| I_N - \left(\frac{f(\varepsilon(t, x + l_{N*})) \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - f(\varepsilon(t, x)) \frac{u_x(t, x)}{|u_x(t, x)|} \right) \right| \\
& \quad + \left| \left(\frac{f(\varepsilon(t, x + l_{N*})) \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - f(\varepsilon(t, x)) \frac{u_x(t, x)}{|u_x(t, x)|} \right) - \frac{\partial}{\partial x} \left(f(\varepsilon(t, x)) \frac{u_x(t, x)}{|u_x(t, x)|} \right) \right| \\
& =: I_{1N} + I_{2N} \quad \text{for } N \in \mathbb{Z}_{>0}.
\end{aligned}$$

By the assumptions for differentiability of ε and $u_x(t, x) \neq 0$, we easily get

$$I_{2N} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Hence, we have to prove the following convergence:

$$I_{1N} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

By elementary calculations, we have

$$\begin{aligned}
I_{1N} & \leq \frac{1}{l_{N*}} \left| f(\varepsilon_N(t, x)) \left\{ \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} - \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} \right\} \right. \\
& \quad \left. - \left\{ f(\varepsilon_N(t, x - l_{N*})) \left\{ \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} + \frac{u_x(t, x)}{|u_x(t, x)|} \right\} \right\} \right| \\
& \quad + \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t, x)) - f(\varepsilon(t, x + l_{N*})) \} \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} \right. \\
& \quad \left. - \{ f(\varepsilon_N(t, x - l_{N*})) - f(\varepsilon(t, x)) \} \frac{u_x(t, x)}{|u_x(t, x)|} \right| + o\left(\frac{1}{N}\right) \frac{N}{l_*}.
\end{aligned}$$

By (3.1.3), we see that $o\left(\frac{1}{N}\right) \frac{N}{l_*} \rightarrow 0$ as $N \rightarrow \infty$. We put

$$\begin{aligned}
J_{1N}^{(1)} & = \frac{1}{l_{N*}} \left| f(\varepsilon_N(t, x)) \left\{ \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} - \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} \right\} \right. \\
& \quad \left. - \left\{ f(\varepsilon_N(t, x - l_{N*})) \left\{ \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} + \frac{u_x(t, x)}{|u_x(t, x)|} \right\} \right\} \right|, \\
J_{1N}^{(2)} & = \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t, x)) - f(\varepsilon(t, x + l_{N*})) \} \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} \right. \\
& \quad \left. - \{ f(\varepsilon_N(t, x - l_{N*})) - f(\varepsilon(t, x)) \} \frac{u_x(t, x)}{|u_x(t, x)|} \right| \quad \text{for } N \in \mathbb{Z}_{>0}.
\end{aligned}$$

2nd step. $J_{1N}^{(1)}(t, x) \rightarrow 0$ as $N \rightarrow \infty$.

Proof of 2nd step. Immediately, we see that

$$J_{1N}^{(1)} \leq \frac{|f(\varepsilon_N(t, x))|}{l_{N*}} |A_{1N}^{(1)} - B_{1N}^{(1)}| + \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t, x)) - f(\varepsilon_N(t, x - l_{N*})) \} B_{1N}^{(1)} \right|, \quad (3.1.5)$$

where

$$\begin{aligned} A_{1N}^{(1)} &= \frac{u(t, x + l_{N*}) - u(t, x)}{l_N(t, x)} - \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|}, \\ B_{1N}^{(1)} &= \frac{u(t, x) - u(t, x - l_{N*})}{l_N(t, x - l_{N*})} + \frac{u_x(t, x)}{|u_x(t, x)|}. \end{aligned}$$

It is clear that

$$\begin{aligned} A_{1N}^{(1)} &= \frac{\frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}}}{\frac{l_N(t, x)}{l_{N*}}} - \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} \\ &= \frac{\frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} \left\{ |u_x(t, x + l_{N*})| - \left| \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} \right| \right\}}{\left| \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} \right| |u_x(t, x + l_{N*})|} \\ &\quad + \frac{\frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} - u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|}. \end{aligned}$$

By putting

$$\begin{aligned} A_{1N}^{(1,1)} &= \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}}, \\ A_{1N}^{(1,2)} &= \left| \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} \right| |u_x(t, x + l_{N*})|, \\ A_{1N}^{(1,3)} &= |u_x(t, x + l_{N*})| - \left| \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} \right|, \\ A_{1N}^{(1,4)} &= \frac{u(t, x + l_{N*}) - u(t, x)}{l_{N*}} - u_x(t, x + l_{N*}), \\ A_{1N}^{(1,5)} &= |u_x(t, x + l_{N*})|, \end{aligned}$$

we have

$$A_{1N}^{(1)} = \frac{A_{1N}^{(1,1)} A_{1N}^{(1,3)}}{A_{1N}^{(1,2)}} + \frac{A_{1N}^{(1,4)}}{A_{1N}^{(1,5)}}.$$

Similarly, we have

$$B_{1N}^{(1)} = \frac{\frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} \left\{ |u_x(t, x)| - \left| \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} \right| \right\}}{\left| \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} \right| |u_x(t, x)|} + \frac{\frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} - u_x(t, x)}{|u_x(t, x)|}.$$

By putting

$$\begin{aligned} B_{1N}^{(1,1)} &= \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}}, \\ B_{1N}^{(1,2)} &= \left| \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} \right| |u_x(t, x)|, \end{aligned}$$

$$\begin{aligned}
B_{1N}^{(1,3)} &= |u_x(t, x)| - \left| \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} \right|, \\
B_{1N}^{(1,4)} &= \frac{u(t, x) - u(t, x - l_{N*})}{l_{N*}} - u_x(t, x), \\
B_{1N}^{(1,5)} &= |u_x(t, x)|,
\end{aligned}$$

we have

$$B_{1N}^{(1)} = \frac{B_{1N}^{(1,1)} B_{1N}^{(1,3)}}{B_{1N}^{(1,2)}} + \frac{B_{1N}^{(1,4)}}{B_{1N}^{(1,5)}}. \quad (3.1.6)$$

Thanks to 1st step, there exist positive constants C_1 and C_2 such that

$$|\varepsilon_N(t, x)| \leq C_1, \quad |f(\varepsilon_N(t, x))| \leq C_2 \quad \text{for } N = 1, 2, \dots \quad (3.1.7)$$

By using (3.1.7), we can estimate the first term in the right hand side of (3.1.5) as follows

$$\begin{aligned}
& \frac{f(\varepsilon_N(t, x))}{l_{N*}} \left| A_{1N}^{(1)} - B_{1N}^{(1)} \right| \\
& \leq \frac{C_2}{l_{N*}} \left| A_{1N}^{(1)} - B_{1N}^{(1)} \right| \\
& \leq C_2 \left| \frac{1}{l_{N*}} \left\{ \frac{A_{1N}^{(1,1)} A_{1N}^{(1,3)}}{A_{1N}^{(1,2)}} + \frac{A_{1N}^{(1,4)}}{A_{1N}^{(1,5)}} \right\} - \frac{1}{l_{N*}} \left\{ \frac{B_{1N}^{(1,1)} B_{1N}^{(1,3)}}{B_{1N}^{(1,2)}} + \frac{B_{1N}^{(1,4)}}{B_{1N}^{(1,5)}} \right\} \right|. \quad (3.1.8)
\end{aligned}$$

By using Taylor's series expansion, we can take $\theta_{1N} \in [0, 1]$ such that

$$u(t, x) = u(t, x + l_{N*}) - l_{N*} u_x(t, x + l_{N*}) + \frac{l_{N*}^2}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}). \quad (3.1.9)$$

We note that

$$A_{1N}^{(1,3)} = |u_x(t, x + l_{N*})| - \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right|,$$

and by putting

$$\begin{aligned}
A_{1N}^{(1,3,1)} &= \sqrt{|u_x(t, x + l_{N*})|^2}, \\
A_{1N}^{(1,3,2,1)} &= (u_x(t, x + l_{N*}), u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})), \\
A_{1N}^{(1,3,2)} &= \sqrt{|u_x(t, x + l_{N*})|^2 - l_{N*} A_{1N}^{(1,3,2,1)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2},
\end{aligned}$$

we have

$$\begin{aligned}
A_{1N}^{(1,3)} &= \frac{(A_{1N}^{(1,3,1)} - A_{1N}^{(1,3,2)}) (A_{1N}^{(1,3,1)} + A_{1N}^{(1,3,2)})}{A_{1N}^{(1,3,1)} + A_{1N}^{(1,3,2)}} \\
&= -\frac{1}{A_{1N}^{(1,3,1)} + A_{1N}^{(1,3,2)}} \left\{ l_{N*} A_{1N}^{(1,3,2,1)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2 \right\}.
\end{aligned}$$

By using this equation, we can deal with the first term in the right hand side of (3.1.8) as follows:

$$\begin{aligned} & \frac{1}{l_{N*}} \frac{A_{1N}^{(1,1)} A_{1N}^{(1,3)}}{A_{1N}^{(1,2)}} \\ &= -\frac{A_{1N}^{(1,1)}}{A_{1N}^{(1,2)}} \frac{1}{A_{1N}^{(1,3,1)} + A_{1N}^{(1,3,2)}} \\ & \quad \times \left\{ (u_x(t, x + l_{N*}), u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})) + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2 \right\}. \end{aligned}$$

It is obvious that

$$\frac{1}{l_{N*}} \frac{A_{1N}^{(1,1)} A_{1N}^{(1,3)}}{A_{1N}^{(1,2)}} \rightarrow -\frac{u_x(t, x)}{2 |u_x(t, x)|^3} (u_x(t, x), u_{xx}(t, x)) \quad \text{as } N \rightarrow \infty. \quad (3.1.10)$$

Similarly, by (3.1.9), we have

$$A_{1N}^{(1,4)} = -\frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}),$$

and

$$\begin{aligned} \frac{1}{l_{N*}} \frac{A_{1N}^{(1,4)}}{A_{1N}^{(1,5)}} &= -\frac{u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})}{2 |u_x(t, x + l_{N*})|} \\ &\rightarrow -\frac{u_{xx}(t, x)}{2 |u_x(t, x)|} \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (3.1.11)$$

Moreover, by a similar argument as above, we obtain the following convergences.

$$\frac{1}{l_{N*}} \frac{B_{1N}^{(1,1)} B_{1N}^{(1,3)}}{B_{1N}^{(1,2)}} \rightarrow -\frac{u_x(t, x)}{2 |u_x(t, x)|^3} (u_x(t, x), u_{xx}(t, x)) \quad \text{as } N \rightarrow \infty, \quad (3.1.12)$$

$$\frac{1}{l_{N*}} \frac{B_{1N}^{(1,4)}}{B_{1N}^{(1,5)}} \rightarrow -\frac{u_{xx}(t, x)}{2 |u_x(t, x)|} \quad \text{as } N \rightarrow \infty. \quad (3.1.13)$$

By using (3.1.10)–(3.1.13), we obtain

$$\begin{aligned} & \frac{|f(\varepsilon_N(t, x))|}{l_{N*}} |A_{1N}^{(1)} - B_{1N}^{(1)}| \\ & \leq C_2 \left| \frac{1}{l_{N*}} \left\{ \frac{A_{1N}^{(1,1)} A_{1N}^{(1,3)}}{A_{1N}^{(1,2)}} + \frac{A_{1N}^{(1,4)}}{A_{1N}^{(1,5)}} \right\} - \frac{1}{l_{N*}} \left\{ \frac{B_{1N}^{(1,1)} B_{1N}^{(1,3)}}{B_{1N}^{(1,2)}} + \frac{B_{1N}^{(1,4)}}{B_{1N}^{(1,5)}} \right\} \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned} \quad (3.1.14)$$

Here, we deal with in the right hand side of (3.1.5) as follows:

$$\begin{aligned} & \frac{1}{l_{N*}} |f(\varepsilon_N(t, x)) - f(\varepsilon_N(t, x - l_{N*}))| |B_{1N}^{(1)}| \\ &= |f(\varepsilon_N(t, x)) - f(\varepsilon(t, x)) + f(\varepsilon(t, x)) - f(\varepsilon_N(t, x - l_{N*}))| \left| \frac{B_{1N}^{(1)}}{l_{N*}} \right|. \end{aligned}$$

By using (3.1.6), (3.1.12) and (3.1.13), we have

$$\left| \frac{B_{1N}^{(1)}}{l_{N*}} \right| \rightarrow \frac{1}{2 |u_x(t, x)|} \left| \frac{u_x(t, x)}{|u_x(t, x)|^2} (u_x(t, x), u_{xx}(t, x)) + u_{xx}(t, x) \right| \quad \text{as } N \rightarrow \infty.$$

Since $f(\varepsilon_N) \rightarrow f(\varepsilon)$ as $N \rightarrow \infty$, it holds that

$$\frac{1}{l_{N*}} |f(\varepsilon_N(t, x)) - f(\varepsilon_N(t, x - l_{N*}))| \left| B_{1N}^{(1)} \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (3.1.15)$$

Hence, by (3.1.14) and (3.1.15), we obtain

$$J_{1N}^{(1)} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

□

3rd step. $J_{1N}^{(2)} \rightarrow 0$ as $N \rightarrow \infty$.

Proof of 3rd step. First, we see that

$$\begin{aligned} J_{1N}^{(2)} &\leq \frac{1}{l_{N*}} |f(\varepsilon_N(t, x)) - f(\varepsilon(t, x + l_{N*}))| \left| \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - \frac{u_x(t, x)}{|u_x(t, x)|} \right| \\ &\quad + \frac{1}{l_{N*}} \left| \{f(\varepsilon_N(t, x)) - f(\varepsilon(t, x + l_{N*})) - f(\varepsilon_N(t, x - l_{N*})) + f(\varepsilon(t, x))\} \frac{u_x(t, x)}{|u_x(t, x)|} \right| \\ &=: J_{1N}^{(2,1)} + J_{1N}^{(2,2)}. \end{aligned}$$

Since, f is locally Lipschitz continuous on $(-1, \infty)$, we can take a positive constant C_f such that

$$|f(\varepsilon_N(t, x)) - f(\varepsilon(t, x + l_{N*}))| \leq C_f |\varepsilon_N(t, x) - \varepsilon(t, x + l_{N*})| \quad \text{for } N \in \mathbb{Z}_{>0}.$$

By using this inequality and (3.1.3), we obtain

$$\begin{aligned} J_{1N}^{(2,1)} &\leq \frac{C_f}{l_{N*}} |\varepsilon_N(t, x) - \varepsilon(t, x + l_{N*})| \left| \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - \frac{u_x(t, x)}{|u_x(t, x)|} \right| \\ &\leq \frac{C_f}{l_{N*}} \left| \frac{1}{l_{N*}} \{u(t, x + l_{N*}) - u(t, x)\} - u_x(t, x + l_{N*}) \right| \left| \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - \frac{u_x(t, x)}{|u_x(t, x)|} \right|. \end{aligned}$$

Here, by using (3.1.9), we have

$$J_{1N}^{(2,1)} \leq \frac{C_f}{2} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})| \left| \frac{u_x(t, x + l_{N*})}{|u_x(t, x + l_{N*})|} - \frac{u_x(t, x)}{|u_x(t, x)|} \right|.$$

Moreover, since u_{xx} is continuous, there exists a positive constant C such that

$$|u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})| \leq C \quad \text{for all } N = 1, 2, \dots \quad (3.1.16)$$

By $u_x \in C(\overline{Q(T)})$ and $u_x \neq 0$ on $\overline{Q(T)}$, we obtain

$$J_{1N}^{(2,1)} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Next, we shall show the convergence of $J_{1N}^{(2,2)}$. By using Taylor's series expansion for $f(\varepsilon_N)$ we can choose $\theta_{2N}, \theta_{3N} \in [0, 1]$ such that

$$f(\varepsilon_N(t, x)) = f(\varepsilon(t, x + l_{N*})) + \{\varepsilon_N(t, x) - \varepsilon(t, x + l_{N*})\} f'(\varepsilon(t, x + l_{N*}) + \theta_{2N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})\}),$$

$$f(\varepsilon_N(t, x - l_{N*})) = f(\varepsilon(t, x)) + \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)\} f'(\varepsilon(t, x) + \theta_{3N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)\}).$$

Here, by using (3.1.3) and (3.1.9), we have

$$\varepsilon_N(t, x) - \varepsilon(t, x + l_{N*}) = \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right| - |u_x(t, x + l_{N*})|.$$

Also, by using Taylor's series expansion for $u(t, x - l_{N*})$, there exists $\theta_{4N} \in [0, 1]$ such that

$$\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x) = \left| u_x(t, x) - \frac{l_{N*}}{2} u_{xx}(t, x - \theta_{4N} l_{N*}) \right| - |u_x(t, x)|.$$

By putting

$$\begin{aligned} C_N &= \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right| - |u_x(t, x + l_{N*})|, \\ D_N &= f'(\varepsilon(t, x + l_{N*}) + \theta_{2N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})\}), \\ E_N &= \left| u_x(t, x) - \frac{l_{N*}}{2} u_{xx}(t, x - \theta_{4N} l_{N*}) \right| - |u_x(t, x)|, \\ F_N &= f'(\varepsilon(t, x) + \theta_{3N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)\}), \end{aligned}$$

we have

$$\begin{aligned} J_{1N}^{(2,2)} &= \frac{1}{l_{N*}} |C_N D_N - E_N F_N| \\ &\leq \frac{1}{l_{N*}} |C_N| |D_N - F_N| + \frac{1}{l_{N*}} |C_N - E_N| |F_N|. \end{aligned}$$

Since f' is locally Lipschitz continuous on $(-1, \infty)$ and $\varepsilon > 0$ on $\overline{Q(T)}$, there exists a positive constant C'_f such that

$$|D_N - F_N| \leq C'_f \{|\varepsilon(t, x + l_{N*}) + \theta_{2N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})\} - \{\varepsilon(t, x) + \theta_{3N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)\}\}|\} \quad \text{for all } N = 1, 2, \dots$$

By using this inequality and (3.1.16), we have

$$\begin{aligned} &\frac{1}{l_{N*}} |C_N| |D_N - F_N| \\ &\leq \frac{C'_f C}{2} \{|\varepsilon(t, x + l_{N*}) - \varepsilon(t, x)| + \theta_{2N} |\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})|\} \end{aligned}$$

$$+\theta_{3N} |\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)|\} \quad \text{for all } N = 1, 2, \dots$$

Thanks to 1st step, we obtain the following convergence:

$$\frac{1}{l_{N*}} |C_N| |D_N - F_N| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Since f' is continuous on $(-1, \infty)$, 1st step guarantees existence of a positive constant C' such that

$$|f'(\varepsilon(t, x) + \theta_{3N} \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x)\})| \leq C' \quad \text{for all } N = 1, 2, \dots$$

On account of this boundedness, we have

$$\begin{aligned} & \frac{1}{l_{N*}} |C_N - E_N| |F_N| \\ & \leq \frac{C'}{l_{N*}} \left| C_N^{(1)} - C_N^{(2)} - (E_N^{(1)} - E_N^{(2)}) \right| \\ & = C' \left| \frac{C_N^{(1,2)} + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2}{C_N^{(1)} + C_N^{(2)}} - \frac{E_N^{(1,2)} + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{4N} l_{N*})|^2}{E_N^{(1)} + E_N^{(2)}} \right|, \end{aligned}$$

where

$$\begin{aligned} C_N^{(1)} &= \sqrt{|u_x(t, x + l_{N*})|^2 + l_{N*} C_N^{(1,2)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2}, \\ C_N^{(1,2)} &= (u_x(t, x + l_{N*}), u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})), \\ C_N^{(2)} &= \sqrt{|u_x(t, x + l_{N*})|^2}, \\ E_N^{(1)} &= \sqrt{|u_x(t, x)|^2 + l_{N*} E_N^{(1,2)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{4N} l_{N*})|^2}, \\ E_N^{(1,2)} &= (u_x(t, x), u_{xx}(t, x - \theta_{4N} l_{N*})), \\ E_N^{(2)} &= \sqrt{|u_x(t, x + l_{N*})|^2}. \end{aligned}$$

Now, the following convergences hold:

$$\begin{aligned} \frac{C_N^{(1,2)} + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2}{C_N^{(1)} + C_N^{(2)}} &\rightarrow \frac{(u_x(t, x), u_{xx}(t, x))}{2 |u_x(t, x)|}, \\ \frac{E_N^{(1,2)} + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{4N} l_{N*})|^2}{E_N^{(1)} + E_N^{(2)}} &\rightarrow \frac{(u_x(t, x), u_{xx}(t, x))}{2 |u_x(t, x)|} \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Thus, by using these convergences, we obtain

$$\frac{1}{l_{N*}} |C_N - E_N| |F_N| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Namely, it holds that $J_{1N}^{(2,2)} \rightarrow 0$ as $N \rightarrow \infty$, and $J_{1N}^{(2)} \rightarrow 0$ as $N \rightarrow \infty$. □

Hence, we have proved (3.1.4). Therefore, we obtain (3.1.1) as the kinetic equation.

It is not easy to solve (3.1.1), since the coefficient of u_x is nonlinear and singular. Accordingly, in order to deal with the equation, mathematically, we add the fourth derivative term to (3.1.1) as follows:

$$\rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\frac{\partial u}{\partial x}}{\left| \frac{\partial u}{\partial x} \right|} \right) = 0, \varepsilon = \left| \frac{\partial u}{\partial x} \right| - 1 \text{ on } Q(T), \quad (3.1.17)$$

The equation (3.1.17) is called a beam equation, and appears when we approximate the motion of a 3-dimensional elastic material by the 1-dimensional model. Also, in [9] the fourth derivative term is regarded as a description of non-local effects as induced by interfacial energy. For simplicity, we put $f_0(\varepsilon) = f(\varepsilon)/(1 + \varepsilon)$ and write f_0 by f . Moreover, we impose the periodic boundary condition (1.3.9) representing the material that is smoothly connected at $x = 0, 1$, and the initial condition (1.3.10). Thus, we obtain the initial and boundary value problem $P_0(u_0, v_0)$ for partial differential equation.

3.2 Mathematical results

Throughout Chapters 3 and 4, we use the following notations for the spaces

$$\begin{aligned} H &:= L^2(0, 1)^2, V_1 := \{z \in H^2(0, 1)^2 | z(0) = z(1), z_x(0) = z_x(1)\}, \\ V_2 &:= \{z \in H^4(0, 1)^2 | z(0) = z(1), z_x(0) = z_x(1), z_{xx}(0) = z_{xx}(1), z_{xxx}(0) = z_{xxx}(1)\}, \end{aligned}$$

with standard norms denoted by $|\cdot|_H = |\cdot|_{L^2(0,1)^2}$, $|\cdot|_{V_1} = |\cdot|_{H^2(0,1)^2}$, $|\cdot|_{V_2} = |\cdot|_{H^4(0,1)^2}$, respectively.

First, we give a definition for a weak solution of P_0 .

Definition 3.1. A function u from $Q(T)$ to \mathbb{R}^2 is called a weak solution of $P_0(u_0, v_0)$ on $Q(T)$ if u has the following properties: $u \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V_1)$, $u(0) = u_0$ and satisfying

$$\begin{aligned} & -\rho \int_{Q(T)} u_t \cdot \eta_t dx dt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dx dt + \int_{Q(T)} f(\varepsilon) u_x \cdot \eta_x dx dt \\ & = \rho \int_0^1 v_0 \cdot \eta(0) dx \text{ for } \eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V_1) \text{ with } \eta(T) = 0. \end{aligned}$$

We note that $u \cdot v = u_1 v_1 + u_2 v_2$ for $u = (u_1, v_1)$, $v = (v_1, v_2) \in \mathbb{R}^2$. The main result of this part is as follows:

Theorem 3.1. *Let $T > 0$. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous, monotone increasing and $f(0) = 0$, $u_0 \in V_1$ and $v_0 \in H$, then $P_0(u_0, v_0)$ has a unique weak solution on $Q(T)$.*

The proof of the uniqueness is given in the next section. In Section 3.4 we prove the existence of the solutions.

3.3 Uniqueness

In this section we give a proof of the uniqueness for the solution to P_0 and suppose that all assumptions of Theorem 3.1 hold.

Let u_1 and u_2 be solutions of P_0 , namely, u_1 and u_2 satisfies the properties of Definition 3.1. Also, we put $u = u_1 - u_2$, and $W = \left\{ \eta \in W^{2,2}(0, T; H) \cap L^2(0, T; H^4(0, 1)^2) \mid \eta(T) = \eta_t(T) = 0, \frac{\partial^i}{\partial x^i} \eta(t, 0) = \frac{\partial^i}{\partial x^i} \eta(t, 1) \text{ for } t \in [0, T] \text{ and } i = 0, 1, 2, 3 \right\}$. For any $\eta \in W$, we have

$$\begin{aligned} & -\rho \int_{Q(T)} u_t \cdot \eta_t dxdt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dxdt \\ & + \int_{Q(T)} \{f(\varepsilon_1)u_{1x} - f(\varepsilon_2)u_{2x}\} \cdot \eta_x dxdt = 0. \end{aligned}$$

By integrating by parts in this equation, we have

$$\begin{aligned} & \rho \int_{Q(T)} u \cdot \eta_{tt} dxdt + \gamma \int_{Q(T)} u \cdot \eta_{xxxx} dxdt \\ & = - \int_{Q(T)} \{(f(\varepsilon_1) - f(\varepsilon_2))u_{1x} + f(\varepsilon_2)(u_{1x} - u_{2x})\} \cdot \eta_x dxdt \\ & = - \int_{Q(T)} \{F_0(a \cdot u_{1x}) + f(\varepsilon_2)\} \eta_x \cdot u_x dxdt, \end{aligned} \tag{3.3.1}$$

where $\varepsilon = \varepsilon_1 - \varepsilon_2$, $a = (a^{(1)}, a^{(2)})$,

$$F_0 = \begin{cases} \frac{f(\varepsilon_1) - f(\varepsilon_2)}{\varepsilon_1 - \varepsilon_2} & \text{if } \varepsilon_1 \neq \varepsilon_2, \\ 0 & \text{if } \varepsilon_1 = \varepsilon_2, \end{cases} \quad a^{(i)} = \begin{cases} \frac{u_{1x}^{(i)} + u_{2x}^{(i)}}{|u_{1x}| + |u_{2x}|} & \text{if } |u_{1x}| + |u_{2x}| \neq 0, \\ 0 & \text{if } |u_{1x}| + |u_{2x}| = 0, \end{cases}$$

for $i = 1, 2$, and $u_{jx} = (u_{jx}^{(1)}, u_{jx}^{(2)})$ for $j = 1, 2$. Also, we put $F = F_0 a \cdot u_{1x} + f(\varepsilon_2)$, and then (3.3.1) is represented by F as follows:

$$\int_{Q(T)} u \cdot (\rho \eta_{tt} + \gamma \eta_{xxxx}) dxdt + \int_{Q(T)} u_x \cdot (F \eta_x) dxdt = 0 \text{ for } \eta \in W. \tag{3.3.2}$$

Since f is Lipschitz continuous and $u_{1x} \in L^\infty(Q(T))$, we have $F \in L^\infty(Q(T))$ and can approximate it by $\{F_n\} \subset C_0^\infty(Q(T))$ satisfying

$$\{F_n\} \text{ is bounded in } L^\infty(Q(T)) \text{ and } F_n \rightarrow F \text{ in } L^2(Q(T)) \text{ as } n \rightarrow \infty. \tag{3.3.3}$$

The first lemma is concerned with the existence of the solution of the approximate dual problem.

Lemma 3.1. *Let $\varphi \in C_0^\infty(Q(T))^2$. For $n \in \mathbb{Z}_{>0}$, there exists a unique solution $\eta_n \in W^{2,\infty}(0, T; H) \cap L^\infty(0, T; H^4(0, 1)^2)$ of the following approximate dual problem:*

$$\rho \eta_{ntt} + \gamma \eta_{nxxxx} - (F_n \eta_{nx})_x = \varphi \text{ in } Q(T), \tag{3.3.4}$$

$$\frac{\partial^i}{\partial x^i} \eta_n(t, 0) = \frac{\partial^i}{\partial x^i} \eta_n(t, 1) \text{ for } t \in [0, T] \text{ and } i = 0, 1, 2, 3, \tag{3.3.5}$$

$$\eta_n(T) = \eta_{nt}(T) = 0 \text{ on } (0, 1). \tag{3.3.6}$$

In the proof of Lemma 3.1, we use the following Lemmas 3.2 and 3.3.

Lemma 3.2. *Let $T > 0$.*

- (1) *Let $n \in \mathbb{Z}_{>0}$ and $h = \frac{T}{n}$. If $u_0 \in V_2$, $v_0 \in V_2$, and $\varphi \in C([0, T]; V_1)$, then there exists a unique solution $u_i^n \in H^4(0, 1)^2$ for $i = 2, 3, \dots, n$ satisfies*

$$\begin{aligned} \frac{\rho}{h^2} u_i^n + \gamma u_{xxxx}^n &= \varphi_i^n + \frac{\rho}{h^2} (2u_{i-1}^n - u_{i-2}^n) \text{ on } (0, 1), \\ \frac{\partial^j}{\partial x^j} u_i^n(0) &= \frac{\partial^j}{\partial x^j} u_i^n(1) \text{ for } j = 0, 1, 2, 3, \\ u_0^n &= u_0, u_1^n = u_0 + hv_0, \end{aligned}$$

where $\varphi_i^n = \varphi(ih)$ for $i = 1, 2, \dots, n$. Moreover, there exists a positive constant C such that

$$|u_i^n|_{H^4(0,1)^2} \leq C, |v_i^n|_{H^2(0,1)^2} \leq C, \left| \frac{v_i^n - v_{i-1}^n}{h} \right|_H \leq C \text{ for any } i = 1, 2, \dots, n \text{ and for any } n \in \mathbb{Z}_{>0},$$

where $v_i^n = \frac{u_i^n - u_{i-1}^n}{h}$ for $i = 1, 2, \dots, n$ and $v_0^n = v_0$.

- (2) *Let $\varphi \in C([0, T]; V_1)$. If $u_0 \in V_2$, $v_0 \in V_2$, then there exists a unique solution $u \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; H^2(0, 1)^2) \cap L^\infty(0, T; H^4(0, 1)^2)$ such that*

$$\begin{aligned} \rho u_{tt} + \gamma u_{xxxx} &= \varphi \text{ a.e. on } Q(T), \\ \frac{\partial^i}{\partial x^i} u(0) &= \frac{\partial^i}{\partial x^i} u(1) \text{ a.e. on } (0, T) \text{ for } i = 0, 1, 2, 3, \\ u(0) &= u_0, u_t(0) = v_0 \text{ a.e. on } (0, 1). \end{aligned}$$

Moreover, the following inequality holds:

$$\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 \leq \rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx}|_H^2 + \frac{2T}{\rho} \int_0^t |\varphi_{xx}(\tau)|_H^2 d\tau \text{ for any } t \in [0, T].$$

The above results also hold, even if $\varphi \in L^2(0, T; V_1)$ and $v_0 \in V_1$.

Lemma 3.3. *If $u_{n0} \in V_2$, $v_{n0} \in V_1$, $\widehat{F}_n \in C_0^\infty(Q(T))$, and $\widehat{\varphi}_n \in L^2(0, T; V_1)$, then there exists a solution $\widehat{\eta}_n \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; H^2(0, 1)^2) \cap L^\infty(0, T; H^4(0, 1)^2)$ satisfies*

$$\rho \widehat{\eta}_{ntt} + \gamma \widehat{\eta}_{nxxxx} - \left(\widehat{F}_n \widehat{\eta}_{nx} \right)_x = \widehat{\varphi} \text{ a.e. on } Q(T), \quad (3.3.7)$$

$$\frac{\partial^i}{\partial x^i} \widehat{\eta}_n(0) = \frac{\partial^i}{\partial x^i} \widehat{\eta}_n(1) \text{ a.e. on } (0, T) \text{ and } i = 0, 1, 2, 3, \quad (3.3.8)$$

$$\widehat{\eta}_n(0) = u_{n0}, \widehat{\eta}_{nt}(0) = v_{n0} \text{ a.e. on } (0, 1). \quad (3.3.9)$$

We give a proof of Lemma 3.3 after we prove Lemma 3.2. In order to prove Lemma 3.2, we prepare the following Lemma 3.4.

Lemma 3.4.

- (1) *If $z \in H$, $z_{xx} \in H$ in the sense of the distribution, then $z_x \in H$.*

(2) If $z \in (C_0^\infty(0,1)^2)'$ and $z_x = 0$ in $(C_0^\infty(0,1)^2)'$, then there exists $c \in \mathbb{R}$ such that

$$\langle c, \varphi \rangle_{C_0^\infty(0,1)^2} = \langle z, \varphi \rangle_{C_0^\infty(0,1)^2} \text{ for any } \varphi \in C_0^\infty(0,1)^2,$$

where $\langle \cdot, \cdot \rangle_{C_0^\infty(0,1)^2}$ denotes the duality pair in $C_0^\infty(0,1)^2$.

Proof of Lemma 3.4. (1) is a direct applications of (2) and the Radon–Nikodym theorem. Hence, it is sufficient to show (2). For a proof of (2), let $z \in (C_0^\infty(0,1)^2)'$ and $z_x = 0$ in $(C_0^\infty(0,1)^2)'$. We can choose $\varphi_0 \in C_0^\infty(0,1)^2$ satisfying $\int_0^1 \varphi_0 dx = 1$. Let $\varphi \in C_0^\infty(0,1)^2$, and put $\lambda = \int_0^1 \varphi dx$ and $\eta = \varphi - \lambda \varphi_0$. Easily, we have

$$\int_0^1 \eta dx = \int_0^1 \varphi dx - \lambda \int_0^1 \varphi_0 dx = \lambda \left(1 - \int_0^1 \varphi_0 dx \right) = 0,$$

and $\eta \in C_0^\infty(0,1)^2$. Put $\psi(x) = \int_0^x \eta(t) dt$, then we see that $\frac{d\psi}{dx} = \eta$ and $\psi \in C_0^\infty(0,1)^2$ hold. By putting $c = \langle z, \varphi_0 \rangle_{C_0^\infty(0,1)^2}$, it is clear that $c \in \mathbb{R}$. Furthermore, since $\eta = \varphi - \lambda \varphi_0$, we have

$$\begin{aligned} \langle z, \varphi \rangle_{C_0^\infty(0,1)^2} &= \langle z, \eta + \lambda \varphi_0 \rangle_{C_0^\infty(0,1)^2} \\ &= \langle z, \eta \rangle_{C_0^\infty(0,1)^2} + \lambda \langle z, \varphi_0 \rangle_{C_0^\infty(0,1)^2} \\ &= \langle z, \eta \rangle_{C_0^\infty(0,1)^2} + c \langle 1, \varphi \rangle_{C_0^\infty(0,1)^2} \\ &= \langle z, \psi_x \rangle_{C_0^\infty(0,1)^2} + \langle c, \varphi \rangle_{C_0^\infty(0,1)^2} \\ &= - \langle z_x, \psi \rangle_{C_0^\infty(0,1)^2} + \langle c, \varphi \rangle_{C_0^\infty(0,1)^2} \text{ for any } \varphi \in C_0^\infty(0,1)^2. \end{aligned}$$

In this calculation, we use $\lambda = \langle 1, \varphi \rangle_{C_0^\infty(0,1)^2}$. On account of $z_x = 0$ in $(C_0^\infty(0,1)^2)'$, we get

$$\langle z, \varphi \rangle_{C_0^\infty(0,1)^2} = \langle c, \varphi \rangle_{C_0^\infty(0,1)^2} \text{ for any } \varphi \in C_0^\infty(0,1)^2.$$

Thus, Lemma 3.4 has been proved. □

Proof of Lemma 3.2. This lemma can be proved in the following 5 steps. Let $T > 0$.

(Step 1) If $h > 0$, and $f \in H$, then there exists $u \in H^4(0,1)^2$ such that

$$\frac{\rho}{h^2} u + \gamma u_{xxxx} = f \text{ a.e. on } (0,1), \quad (3.3.10)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \text{ a.e. on } (0,T) \text{ for } i = 0, 1, 2, 3. \quad (3.3.11)$$

This step will be proved by the Riesz representation theorem. In order to prove the step we define weak solutions of (3.3.10)–(3.3.11) as follows.

Definition of weak solutions of (3.3.10)–(3.3.11). A function u on $(0,1)$ is called a weak solution of (3.3.10)–(3.3.11) on $(0,1)$ if u satisfies that $u \in V_1$ and

$$\frac{\rho}{h^2} \int_0^1 u(x) \cdot z(x) dx + \gamma \int_0^1 u_{xx}(x) \cdot z_{xx}(x) dx = \int_0^1 f(x) \cdot z(x) dx \text{ for any } z \in V_1. \quad (3.3.12)$$

We show existence of a weak solution of (3.3.10)–(3.3.11). First, we put

$$\langle L, z \rangle = \int_0^1 f(x) \cdot z(x) dx \text{ for any } z \in H.$$

Clearly, L is the element of the dual space of V_1 . Since the left hand side of (3.3.12) is regarded as the inner product of V_1 , by applying the Riesz representation theorem, for any $f \in H$ there exists only one $u \in V_1$ satisfying $\langle L, z \rangle = (u, z)_{V_1}$ for any $z \in V_1$, namely, $(u, z)_{V_1} = \int_0^1 f(x) \cdot z(x) dx$ for any $z \in V_1$. Thus, the existence and uniqueness of the weak solution of (3.3.10)–(3.3.11) are proved.

Next, we show that a weak solution of (3.3.10)–(3.3.11) is also its strong solution. We define the strong solution as follows.

Definition of strong solutions of (3.3.10)–(3.3.11). *A function u on $(0, 1)$ is called a strong solution of (3.3.10)–(3.3.11) on $(0, 1)$ if u satisfies that $u \in H^4(0, 1)^2$ and satisfies (3.3.10) and (3.3.11) in the usual sense.*

Let u be a weak solution of (3.3.10)–(3.3.11), namely, it holds that

$$(u, z)_{V_1} = \int_0^1 f(x) \cdot z(x) dx \text{ for any } z \in V_1,$$

and

$$\frac{\rho}{h^2} \int_0^1 u(x) \cdot z(x) dx + \gamma \int_0^1 u_{xx}(x) \cdot z_{xx}(x) dx = \int_0^1 f(x) \cdot z(x) dx \text{ for any } z \in V_1. \quad (3.3.13)$$

By (3.3.13), it is easy to see that

$$\begin{aligned} \langle u_{xxxx}, z \rangle_{C_0^\infty(0,1)^2} &= \langle u_{xx}, z_{xx} \rangle_{C_0^\infty(0,1)^2} \\ &= \int_0^1 u_{xx}(x) \cdot z_{xx}(x) dx \\ &= \frac{1}{\gamma} \left(\int_0^1 \left(f(x) - \frac{\rho}{h^2} u(x) \right) \cdot z(x) dx \right) \text{ for any } z \in C_0^\infty(0, 1)^2. \end{aligned}$$

Therefore, $u_{xxxx} \in H$ and $u_{xxxx} = \frac{1}{\gamma} \left(f - \frac{\rho}{h^2} u \right)$ a.e. on $(0, 1)$. Now, $u \in V_1$ implies $u_{xx} \in H$.

Accordingly, by using Lemma 3.4, we see that $u_{xxx} \in H$. Thus, $u \in H^4(0, 1)^2$.

Next, we show (3.3.11). Here, thanks to $u \in V_1$, (3.3.11) holds for $i = 0, 1$. Hence, we prove (3.3.11) for $i = 2, 3$. Now, by applying integration by parts twice, we infer that

$$\begin{aligned} &\gamma \int_0^1 u_{xx}(x) \cdot z_{xx}(x) dx \\ &= \gamma (u_{xx}(1) \cdot z_x(1) - u_{xx}(0) \cdot z_x(0)) - \gamma \int_0^1 u_{xxx}(x) \cdot z_x(x) dx \\ &= \gamma z_x(0) (u_{xx}(1) - u_{xx}(0)) - \gamma (u_{xxx}(1) \cdot z(1) - u_{xxx}(0) \cdot z(0)) + \gamma \int_0^1 u_{xxxx}(x) \cdot z(x) dx \end{aligned}$$

$$= \gamma z_x(0) (u_{xx}(1) - u_{xx}(0)) - \gamma z(0) (u_{xxx}(1) - u_{xxx}(0)) + \gamma \int_0^1 u_{xxxx}(x) \cdot z(x) dx \text{ for any } z \in V_1.$$

By using (3.3.13) to the left hand side of the above equation, we have

$$\begin{aligned} & \int_0^1 \left(f(x) - \frac{\rho}{h^2} u(x) \right) \cdot z(x) dx \\ &= \gamma z_x(0) (u_{xx}(1) - u_{xx}(0)) - \gamma z(0) (u_{xxx}(1) - u_{xxx}(0)) + \gamma \int_0^1 u_{xxxx}(x) \cdot z(x) dx, \end{aligned}$$

and

$$\gamma z_x(0) (u_{xx}(1) - u_{xx}(0)) = \gamma z(0) (u_{xxx}(1) - u_{xxx}(0)) \text{ for any } z \in V_1. \quad (3.3.14)$$

Now, we put $z = (z^{(1)}, z^{(2)})$, $z_x = (z_x^{(1)}, z_x^{(2)})$, and similarly for u_{xx} and u_{xxx} . Moreover, we put $z^{(1)}(x) = \cos(2\pi x)$ and $z^{(2)}(x) = 0$ for any $x \in (0, 1)$. Obviously, $z \in V_1$. Since, $z^{(1)}(0) = 1$, $z_x^{(1)}(0) = 0$, by (3.3.14) we have $u_{xxx}^{(1)}(1) = u_{xxx}^{(1)}(0)$. By putting $z^{(1)}(x) = 0$ and $z^{(2)}(x) = \cos(2\pi x)$ for $x \in (0, 1)$, we can obtain $u_{xxx}^{(2)}(1) = u_{xxx}^{(2)}(0)$ and $u_{xxx}(0) = u_{xxx}(1)$. In the same way, we can prove $u_{xx}(0) = u_{xx}(1)$. Therefore, (3.3.11) holds, and thus Step 1 has been proved.

(Step 2) Lemma 3.2 (1) holds.

In order to prove Step 2, let $n \in \mathbb{Z}_{>0}$ and $h_n = \frac{T}{n}$. If $u_0 \in V_2$, $v_0 \in V_2$, and $\varphi \in C([0, T]; V_1)$, then by Step 1, there exists a unique solution $u_i^n \in H^4(0, 1)^2$ for $i = 2, 3, \dots, n$ satisfies

$$\frac{\rho}{h_n^2} u_i^n + \gamma u_{ixxxx}^n = \varphi_i^n + \frac{\rho}{h_n^2} (2u_{i-1}^n - u_{i-2}^n) \text{ on } (0, 1), \quad (3.3.15)$$

$$\frac{\partial^j}{\partial x^j} u_i^n(0) = \frac{\partial^j}{\partial x^j} u_i^n(1) \text{ for } j = 0, 1, 2, 3, \quad (3.3.16)$$

$$u_0^n = u_0, u_1^n = u_0 + h_n v_0, \quad (3.3.17)$$

where $\varphi_i^n = \varphi(ih_n)$ for $i = 1, 2, \dots, n$. In Steps 2 and 3, for simplicity, let $h_n = h$. First, we show uniform estimates for u_i^n, v_i^n for $i = 1, 2, \dots, n$. Let $i = 2, 3, \dots, n$. Multiplying both sides of (3.3.15) by $\frac{u_i^n - u_{i-1}^n}{h} = v_i^n$ and integrating it on $(0, 1)$, we see that

$$\begin{aligned} & \frac{\rho}{h^2} \int_0^1 (v_i^n - v_{i-1}^n) \cdot v_i^n dx + \gamma \int_0^1 u_{ixxxx}^n \cdot v_i^n dx \\ &= \int_0^1 \varphi_i^n \cdot v_i^n dx \text{ for any } i = 2, 3, \dots, n, \text{ and } n \in \mathbb{Z}_{>0}. \end{aligned} \quad (3.3.18)$$

Now, by applying integration by parts twice to the second term of the left hand side of (3.3.18), and using (3.3.16), we have

$$\begin{aligned} \gamma \int_0^1 u_{ixxxx}^n \cdot v_i^n dx &= \gamma \int_0^1 u_{ixx}^n \cdot v_{ixx}^n dx \\ &= \frac{\gamma}{h} \int_0^1 u_{ixx}^n (u_{ixx}^n - u_{(i-1)xx}^n) dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\gamma}{h} \left(|u_{ixx}^n|_H^2 - |u_{ixx}^n|_H |u_{(i-1)xx}^n|_H \right) \\
&\geq \frac{\gamma}{2h} \left(|u_{ixx}^n|_H^2 - |u_{(i-1)xx}^n|_H^2 \right) \text{ for any } i = 2, 3, \dots, n, \text{ and } n \in \mathbb{Z}_{>0}.
\end{aligned}$$

By using the Young inequality to (3.3.18), we obtain

$$\frac{\rho}{2h} (|v_i^n|_H^2 - |v_{i-1}^n|_H^2) + \frac{\gamma}{2h} \left(|u_{ixx}^n|_H^2 - |u_{(i-1)xx}^n|_H^2 \right) \leq \frac{\rho}{4T} |\varphi_i^n|_H^2 + \frac{T}{\rho} |v_i^n|_H^2,$$

namely,

$$\frac{1}{2h} (\rho |v_i^n|_H^2 + \gamma |u_{ixx}^n|_H^2) \leq \frac{1}{2h} (\rho |v_{i-1}^n|_H^2 + \gamma |u_{(i-1)xx}^n|_H^2) + \frac{\rho}{4T} |\varphi_i^n|_H^2 + \frac{T}{\rho} |v_i^n|_H^2 \quad (3.3.19)$$

for any $i = 2, 3, \dots, n$, and $n \in \mathbb{Z}_{>0}$.

Taking sum of both sides of (3.3.19) with respect to $i = 1, 2, \dots, k$ for $2 \leq k \leq n$, we have

$$\frac{1}{2h} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \leq \sum_{i=2}^k \left(\frac{T}{\rho} |\varphi_i^n|_H^2 + \frac{\rho}{4T} |v_i^n|_H^2 \right) + \frac{1}{2h} (\rho |v_1^n|_H^2 + \gamma |u_{1xx}^n|_H^2)$$

for any $k = 2, 3, \dots, n$. Since $v_1^n = v_0$, $u_{1xx}^n = u_{0xx} + hv_{0xx}$, we have

$$\frac{1}{2h} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \leq \sum_{i=2}^k \left(\frac{T}{\rho} |\varphi_i^n|_H^2 + \frac{\rho}{4T} |v_i^n|_H^2 \right) + \frac{1}{2h} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2) \quad (3.3.20)$$

for any $k = 2, 3, \dots, n$. Multiplying both sides of (3.3.20) by $h > 0$, we infer that

$$\begin{aligned}
\frac{1}{2} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) &\leq h \sum_{i=2}^k \left(\frac{T}{\rho} |\varphi_i^n|_H^2 + \frac{\rho}{4T} |v_i^n|_H^2 \right) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2) \\
&\leq h \sum_{i=2}^n \left(\frac{T}{\rho} |\varphi_i^n|_H^2 + \frac{\rho}{4T} |v_i^n|_H^2 \right) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2)
\end{aligned}$$

for any $k = 2, 3, \dots, n$. Here, it is easy to see that

$$\begin{aligned}
&\max_{2 \leq k \leq n} \left(\frac{1}{2} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \right) \\
&\leq h \sum_{i=2}^n \left(\frac{T}{\rho} |\varphi_i^n|_H^2 + \frac{\rho}{4T} |v_i^n|_H^2 \right) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2) \\
&\leq \frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 + \frac{h}{2T} \sum_{i=2}^n \frac{1}{2} (\rho |v_i^n|_H^2 + \gamma |u_{ixx}^n|_H^2) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2) \\
&\leq \frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 + \frac{hn}{2T} \max_{2 \leq i \leq n} \left(\frac{1}{2} (\rho |v_i^n|_H^2 + \gamma |u_{ixx}^n|_H^2) \right) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2) \\
&= \frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 + \frac{1}{2} \max_{2 \leq i \leq n} \left(\frac{1}{2} (\rho |v_i^n|_H^2 + \gamma |u_{ixx}^n|_H^2) \right) + \frac{1}{2} (\rho |v_0|_H^2 + \gamma |u_{0xxx} + hv_{0xx}|_H^2).
\end{aligned}$$

Thus, we have

$$\frac{1}{2} \max_{2 \leq k \leq n} \left(\frac{1}{2} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \right) \leq \frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 + (\rho |v_0|_H^2 + \gamma |u_{0xxxx} + hv_{0xx}|_H^2),$$

namely,

$$\frac{1}{4} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \leq \frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 + (\rho |v_0|_H^2 + \gamma |u_{0xxxx} + hv_{0xx}|_H^2)$$

for any $k = 2, 3, \dots, n$. Since $\varphi \in C([0, T]; V_1)$, there exists a positive constant C_φ such that $|\varphi_i^n| \leq C_\varphi$ on $(0, 1)$ for $i = 1, 2, \dots, n$ and it holds that

$$\frac{Th}{\rho} \sum_{i=2}^n |\varphi_i^n|_H^2 \leq \frac{Th C_\varphi^2 n}{\rho} = \frac{T^2 C_\varphi^2}{\rho}.$$

Therefore, we obtain

$$\frac{1}{4} (\rho |v_k^n|_H^2 + \gamma |u_{kxx}^n|_H^2) \leq \frac{T^2 C_\varphi^2}{\rho} + (\rho |v_0|_H^2 + \gamma |u_{0xxxx} + hv_{0xx}|_H^2)$$

for any $k = 2, 3, \dots, n$. Thus, we can obtain the uniform estimates of v_i^n and u_{ixx}^n in H for $i = 1, 2, \dots, n$. Next, similarly to the above argument, by multiplying both sides of (3.3.15) by $-\frac{u_{ixx}^n - u_{(i-1)xx}^n}{h} = -v_{ixx}^n$ and by $\frac{u_{ixxxx}^n - u_{(i-1)xxxx}^n}{h} = v_{ixxxx}^n$, we obtain the uniform estimates for v_{ix}^n and u_{ixxx}^n , v_{ixx}^n and u_{ixxxx}^n in H for $i = 1, 2, \dots, n$, respectively.

By using $u_i^n = hv_i^n + u_{i-1}^n$ and the uniform estimate for v_i^n in H , we can obtain the uniform estimate to u_i^n in H . In fact, for any $i = 2, 3, \dots, n$, thanks to $u_i^n = h \sum_{k=1}^i v_k^n + u_0$, we have

$$|u_i^n|_H \leq h \sum_{k=1}^i |v_k^n|_H + |u_0|_H \text{ for } i = 2, 3, \dots, n.$$

The uniform estimate of u_{ix}^n in H is obtained by using the uniform estimates of u_i^n and u_{ixx}^n in H . In fact, for any $i = 1, 2, \dots, n$,

$$\begin{aligned} |u_{ix}^n|_H^2 &= \int_0^1 u_{ix}^n \cdot u_{ix}^n dx \\ &= - \int_0^1 u_i^n \cdot u_{ixx}^n dx \\ &\leq |u_i^n|_H |u_{ixx}^n|_H. \end{aligned}$$

Thus, there exists a positive constant C such that

$$|u_i^n|_{H^4(0,1)^2} \leq C, |v_i^n|_{H^2(0,1)^2} \leq C \text{ for any } i = 1, 2, \dots, n \text{ and for any } n \in \mathbb{Z}_{>0}.$$

Moreover, from (3.3.15), the uniform estimate to u_{ixxxx}^n in H for $i = 1, 2, \dots, n$ and $\varphi \in C([0, T]; V_1)$, we can get easily a uniform estimate of $\frac{v_i^n - v_{i-1}^n}{h}$ in H for $i = 1, 2, \dots, n$. Thus,

(1) in Lemma 3.2 holds.

(Step 3) Let $n \in \mathbb{Z}_{>0}$, $h = \frac{T}{n}$ and $i = 1, 2, \dots, n$. First, we construct functions u_n and v_n on $Q(T)$ by using the solutions u_i^n and v_i^n of (3.3.15)–(3.3.17) as follows:

$$u_n(t) = \begin{cases} u_0 & \text{for } t = 0, \\ u_{i-1}^n + (t - (i-1)h)v_i^n & \text{for } (i-1)h < t \leq ih, \end{cases}$$

and

$$v_n(t) = \begin{cases} v_0 & \text{for } t = 0, \\ \frac{v_i^n - v_{i-1}^n}{h}(t - (i-1)h) + v_{i-1}^n & \text{for } (i-1)h < t \leq ih. \end{cases}$$

The assertion of this step is show existence of a subsequence $\{n_j\} \subset \{n\}$ and $u \in W^{2,2}(0, T; H) \cap W^{1,2}(0, T; H^2(0, 1)^2) \cap L^2(0, T; H^4(0, 1)^2)$ such that

$$\begin{aligned} u_{n_j} &\rightarrow u \text{ weakly in } L^2(0, T; H^4(0, 1)^2), v_{n_j} \rightarrow u_t \text{ weakly in } L^2(0, T; H^2(0, 1)^2), \\ v_{n_j t} &\rightarrow u_{tt} \text{ weakly in } L^2(0, T; H) \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Easily, u_n and v_n are differentiable a.e. on $Q(T)$ with respect to $t \in (0, T)$ and we see that

$$u_{nt} = v_i^n \text{ and } v_{nt} = \frac{v_i^n - v_{i-1}^n}{h} \text{ on } ((i-1)h, ih) \text{ for } i = 1, 2, \dots, n.$$

Since, $v_i^n = \frac{u_i^n - u_{i-1}^n}{h}$ for $i = 1, 2, \dots, n$, we have

$$\begin{aligned} u_n(t) &= u_{i-1}^n + (t - (i-1)h)v_i^n \\ &= u_{i-1}^n + (t - (i-1)h)\frac{u_i^n - u_{i-1}^n}{h} \\ &= \left(\frac{t}{h} - (i-1)\right)u_i^n + \left(i - \frac{t}{h}\right)u_{i-1}^n \text{ for any } n \in \mathbb{Z}_{>0} \text{ and } (i-1)h < t \leq ih. \end{aligned}$$

Here, by putting $\tau = \frac{t}{h} - (i-1)$, we have

$$u_n(\tau) = \tau u_i^n + (1 - \tau)u_{i-1}^n \text{ for } 0 < \tau \leq 1.$$

Similarly, we can represent v_n by using $\tau \in (0, 1]$ such that

$$v_n(t) = \tau v_i^n + (1 - \tau)v_{i-1}^n \text{ for } 0 < \tau \leq 1.$$

From Step 2, we obtain the uniform estimates of u_i^n and v_i^n , namely, there exists a positive constant C such that

$$|u_i^n|_{H^4(0,1)^2}^n \leq C, |v_i^n|_{H^2(0,1)^2} \leq C, \left| \frac{v_i^n - v_{i-1}^n}{h} \right|_H \leq C \text{ for any } i = 1, 2, \dots, n \text{ and for any } n \in \mathbb{Z}_{>0}. \quad (3.3.21)$$

The estimates show that for any $t \in [0, T]$, the sets $\{u_n(t)\}$ and $\{v_n(t)\}$ are bounded in $H^4(0, 1)^2$ and $H^2(0, 1)^2$, respectively. Also, for a.e. $t \in (0, T)$, $\{v_{nt}(t)\}$ is bounded in H . Accordingly, there exists a subsequence $\{n_j\} \subset \{n\}$, $u \in L^2(0, T; H^4(0, 1)^2)$, $v \in L^2(0, T; H^2(0, 1)^2)$ and $w \in L^2(0, T; H)$ such that

$$\begin{aligned} u_{n_j} &\rightarrow u \text{ weakly in } L^2(0, T; H^4(0, 1)^2), v_{n_j} \rightarrow v \text{ weakly in } L^2(0, T; H^2(0, 1)^2), \\ v_{n_j t} &\rightarrow w \text{ weakly in } L^2(0, T; H) \text{ as } j \rightarrow \infty. \end{aligned}$$

Moreover, it is easily seen that $v_t = w$ and $u_t = v$ a.e. on $Q(T)$. Namely, we have $u \in W^{2,2}(0, T; H) \cap W^{1,2}(0, T; H^2(0, 1)^2) \cap L^2(0, T; H^4(0, 1)^2)$ satisfying

$$\begin{aligned} u_{n_j} &\rightarrow u \text{ weakly in } L^2(0, T; H^4(0, 1)^2), v_{n_j} \rightarrow u_t \text{ weakly in } L^2(0, T; H^2(0, 1)^2), \\ v_{n_j t} &\rightarrow u_{tt} \text{ weakly in } L^2(0, T; H) \text{ as } j \rightarrow \infty. \end{aligned} \quad (3.3.22)$$

This is a conclusion of Step 3.

(Step 4) If $\varphi \in C([0, T]; V_1)$, $u_0 \in V_2$ and $v_0 \in V_2$, then there exists $u \in W^{2,2}(0, T; H) \cap W^{1,2}(0, T; H^2(0, 1)^2) \cap L^2(0, T; H^4(0, 1)^2)$ such that

$$\rho u_{tt} + \gamma u_{xxxx} = \varphi \text{ a.e. on } Q(T), \quad (3.3.23)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \text{ a.e. on } (0, T) \text{ for } i = 0, 1, 2, 3, \quad (3.3.24)$$

$$u(0) = u_0, u_t(0) = v_0 \text{ a.e. on } (0, 1). \quad (3.3.25)$$

In order to show Step 4, we put

$$\varphi_n(t) = \begin{cases} \varphi(0) & \text{for } t = 0, \\ \varphi(ih) & \text{for } (i-1)h < t \leq ih, \ i = 1, 2, \dots, n, \end{cases}$$

where $h = \frac{T}{n}$. Clearly, we have $\varphi_n \rightarrow \varphi$ in $L^2(0, T; V_1)$ as $n \rightarrow \infty$, since $\varphi \in C([0, T]; V_1)$.

By (3.3.15),

$$\frac{\rho}{h} \left(\frac{u_i^n - u_{i-1}^n}{h} - \frac{u_{i-1}^n - u_{i-2}^n}{h} \right) + \gamma u_{xxxx}^n = \varphi_i^n,$$

namely,

$$\rho \frac{v_i^n - v_{i-1}^n}{h} + \gamma u_{xxxx}^n = \varphi_i^n \text{ a.e. on } (0, 1).$$

As mentioned above, $v_{nt} = \frac{v_i^n - v_{i-1}^n}{h}$ and $\varphi_i^n = \varphi_n$ on $((i-1)h, ih) \times (0, 1)$ for $i = 1, 2, \dots, n$. It yields that

$$\rho v_{nt} + \gamma u_{xxxx}^n = \varphi_n \text{ a.e. on } Q(T). \quad (3.3.26)$$

Now, we shall prove the following convergences: For any $\eta \in C_0^\infty(Q(T))^2$,

$$\int_{Q(T)} v_{n_j} \cdot \eta dx dt \rightarrow \int_{Q(T)} u_{tt} \cdot \eta dx dt \text{ as } j \rightarrow \infty, \quad (3.3.27)$$

$$\int_{Q(T)} u_{n_jxxxx} \cdot \eta dxdt \rightarrow \int_{Q(T)} u_{xxxx} \cdot \eta dxdt \text{ as } j \rightarrow \infty, \quad (3.3.28)$$

$$\lim_{j \rightarrow \infty} \int_{Q(T)} u_{n_jxxxx} \cdot \eta dxdt = \lim_{j \rightarrow \infty} \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 u_{ixxxx}^{n_j} \cdot \eta dxdt, \quad (3.3.29)$$

$$\int_{Q(T)} \varphi_{n_j} \cdot \eta dxdt \rightarrow \int_{Q(T)} \varphi \cdot \eta dxdt \text{ as } j \rightarrow \infty. \quad (3.3.30)$$

We can easily show (3.3.27), (3.3.28) and (3.3.30), by using (3.3.22) and $\varphi_n \rightarrow \varphi$ in $L^2(0, T; V_1)$ as $n \rightarrow \infty$. So, we prove (3.3.29). First, we see that

$$\begin{aligned} & \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dxdt - \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 u_{ixxxx}^{n_j}(x) \cdot \eta(t, x) dxdt \\ &= \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 (u_{n_jxxxx}(t, x) - u_{ixxxx}^{n_j}(x)) \cdot \eta(t, x) dxdt. \end{aligned} \quad (3.3.31)$$

We recall $u_n(t) = \left(\frac{t}{h} - (i-1)\right) u_{ixxxx}^n + \left(i - \frac{t}{h}\right) u_{(i-1)xxxx}^n$ on $((i-1)h, ih)$ for $i = 1, 2, \dots, n$. By substituting it to (3.3.31), we have

$$\begin{aligned} & \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dxdt - \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 u_{ixxxx}^{n_j}(x) \cdot \eta(t, x) dxdt \\ &= \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 \left\{ \left(\frac{t}{h} - (i-1)\right) u_{ixxxx}^{n_j} + \left(i - \frac{t}{h}\right) u_{(i-1)xxxx}^{n_j} - u_{ixxxx}^{n_j}(x) \right\} \cdot \eta(t, x) dxdt \\ &= \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 \left(\frac{t}{h} - i\right) (u_{ixxxx}^{n_j}(x) - u_{(i-1)xxxx}^{n_j}) \cdot \eta(t, x) dxdt \text{ for any } j \in \mathbb{Z}_{>0}. \end{aligned}$$

Here, let C_η be a positive constant such that $|\eta| \leq C_\eta$ on $Q(T)$. Thanks to $\left|\frac{t}{h} - i\right| \leq 1$, we infer that

$$\begin{aligned} & \left| \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dxdt - \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 u_{ixxxx}^{n_j}(x) \cdot \eta(t, x) dxdt \right| \\ &= \left| \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 \left(\frac{t}{h} - i\right) (u_{ixxxx}^{n_j}(x) - u_{(i-1)xxxx}^{n_j}) \cdot \eta(t, x) dxdt \right| \\ &\leq \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 |u_{ixxxx}^{n_j}(x) - u_{(i-1)xxxx}^{n_j}| |\eta(t, x)| dxdt \\ &\leq C_\eta \sum_{i=1}^{n_j} \int_{(i-1)h}^{ih} \int_0^1 |u_{ixxxx}^{n_j}(x) - u_{(i-1)xxxx}^{n_j}| dxdt. \end{aligned} \quad (3.3.32)$$

Easily, from (3.3.15), we can get

$$u_{xxxx}^{n_j} - u_{(i-1)xxxx}^{n_j} = \frac{1}{\gamma} (\varphi_i^{n_j} - \varphi_{i-1}^{n_j}) - \frac{h_{n_j}}{\gamma\rho} (v_i^{n_j} - 2v_{i-1}^{n_j} + v_{i-2}^{n_j})$$

on $((i-1)h, ih) \times (0, 1)$ for $i = 1, 2, \dots, n_j$,

(3.3.33)

where $h_n = \frac{T}{n}$. By substituting (3.3.33) into (3.3.32), we have

$$\begin{aligned} & \left| \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dx dt - \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 u_{xxxx}^{n_j}(x) \cdot \eta(t, x) dx dt \right| \\ & \leq C_\eta \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 \left| \frac{1}{\gamma} (\varphi_i^{n_j} - \varphi_{i-1}^{n_j}) - \frac{h_{n_j}}{\gamma\rho} (v_i^{n_j} - 2v_{i-1}^{n_j} + v_{i-2}^{n_j}) \right| dx dt \\ & \leq C_\eta \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 \left| \frac{1}{\gamma} (\varphi(ih_{n_j}) - \varphi((i-1)h_{n_j})) - \frac{h_{n_j}}{\gamma\rho} (v_i^{n_j} - 2v_{i-1}^{n_j} + v_{i-2}^{n_j}) \right| dx dt \\ & \leq C_\eta \sum_{i=1}^{n_j} \left\{ \frac{h_{n_j}}{\gamma} |\varphi(ih_{n_j}) - \varphi((i-1)h_{n_j})|_H^2 + \frac{h_{n_j}}{\gamma\rho} \int_{(i-1)h_{n_j}}^{ih_{n_j}} |v_i^{n_j}|_H^2 dt \right. \\ & \quad \left. + \frac{2h_{n_j}}{\gamma\rho} \int_{(i-1)h_{n_j}}^{ih_{n_j}} |v_{i-1}^{n_j}|_H^2 dt + \frac{h_{n_j}}{\gamma\rho} \int_{(i-1)h_{n_j}}^{ih_{n_j}} |v_{i-2}^{n_j}|_H^2 dt \right\}. \end{aligned}$$

By using (3.3.21), we have

$$\begin{aligned} & \left| \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dx dt - \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 u_{xxxx}^{n_j}(x) \cdot \eta(t, x) dx dt \right| \\ & \leq C_\eta \sum_{i=1}^{n_j} \left\{ \frac{h_{n_j}}{\gamma} |\varphi(ih_{n_j}) - \varphi((i-1)h_{n_j})|_H^2 + \frac{4C^2}{\gamma\rho} h_{n_j}^2 \right\} \\ & = \frac{C_\eta h_{n_j}}{\gamma} \sum_{i=1}^{n_j} |\varphi(ih_{n_j}) - \varphi((i-1)h_{n_j})|_H^2 + \frac{4C^2 C_\eta T^2}{\gamma\rho} \frac{1}{n_j}. \end{aligned}$$

From the uniform continuity of φ in H on $[0, T]$, it follows that

$$\begin{aligned} & \left| \int_{Q(T)} u_{n_jxxxx}(t, x) \cdot \eta(t, x) dx dt - \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 u_{xxxx}^{n_j}(x) \cdot \eta(t, x) dx dt \right| \\ & \leq \frac{C_\eta T}{\gamma} \frac{1}{n_j} \sum_{i=1}^{n_j} |\varphi(ih_{n_j}) - \varphi((i-1)h_{n_j})|_H^2 + \frac{4C^2 C_\eta T^2}{\gamma\rho} \frac{1}{n_j} \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Hence, (3.3.29) has been proved.

The above convergences imply (3.3.23). Indeed, we can take $j \rightarrow \infty$ in (3.3.26), namely, it holds that

$$\lim_{j \rightarrow \infty} \int_{Q(T)} \varphi_{n_j}(t, x) \cdot \eta(t, x) dx dt$$

$$= \rho \lim_{j \rightarrow \infty} \int_{Q(T)} v_{n_j t}(t, x) \cdot \eta(t, x) dx dt + \gamma \lim_{j \rightarrow \infty} \sum_{i=1}^{n_j} \int_{(i-1)h_{n_j}}^{ih_{n_j}} \int_0^1 u_{i xxxx}^{n_j}(x) \cdot \eta(t, x) dx dt. \quad (3.3.34)$$

By combining (3.3.27)–(3.3.30) to (3.3.34), we conclude that

$$\int_{Q(T)} \varphi(t, x) \cdot \eta(t, x) dx dt = \int_{Q(T)} (\rho u_{tt}(t, x) + \gamma u_{xxxx}(t, x)) \eta(t, x) dx dt \text{ for any } \eta \in C_0^\infty(Q(T))^2.$$

Thus, we obtain

$$\rho u_{tt} + \gamma u_{xxxx} = \varphi \text{ a.e. on } Q(T).$$

We can easily show that u satisfies the boundary condition by using (3.3.16) and (3.3.22). In order to prove that u satisfies the initial condition, let $\eta \in C_0^\infty(0, 1)^2$. By (3.3.17) and $u_{n_j}(0) = u_0$ a.e. on $(0, 1)$, we have

$$\left| \int_0^1 (u(0, x) - u_0(x)) \cdot \eta(x) dx \right| = \left| \int_0^1 u(0, x) \cdot \eta(x) dx - \int_0^1 u_{n_j}(0, x) \cdot \eta(x) dx \right| \text{ for any } j \in \mathbb{Z}_{>0}. \quad (3.3.35)$$

The following convergence is true:

$$u_{n_j} \rightarrow u \text{ weakly in } W^{1,2}(0, T; H) \text{ as } j \rightarrow \infty. \quad (3.3.36)$$

Indeed, we recall $\{u_n(t)\}$ is bounded in H for any $t \in [0, T]$. Also, $\{v_i^n\}$ is bounded in $L^\infty(0, T; H)$, and $u_{nt} = v_i^n$ on $((i-1)h, ih)$ for $i = 1, 2, \dots, n$. $\{u_{nt}\}$ is bounded in $L^2(0, T; H)$. Obviously, $\{u_n\}$ is bounded in $W^{1,2}(0, T; H)$. Hence, there exists a subsequence $\{j_k\} \subset \{j\}$ and $s \in W^{1,2}(0, T; H)$ such that

$$u_{j_k} \rightarrow s \text{ weakly in } W^{1,2}(0, T; H) \text{ as } k \rightarrow \infty,$$

where $u_j := u_{n_j}$ for any $j \in \mathbb{Z}_{>0}$. Here, we note that $u_{n_j} \rightarrow u$ weakly in $L^2(0, T; H)$. This means that $s = u$ in $L^2(0, T; H)$, and $W^{1,2}(0, T; H)$. Hence, we get (3.3.36). By (3.3.36) and (3.3.35), we can prove that u satisfies the initial condition. Thus, Step 4 is proved.

(Step 5) If $u \in W^{2,2}(0, T; H) \cap W^{1,2}(0, T; H^2(0, 1)^2) \cap L^2(0, T; H^4(0, 1)^2)$ is a solution of the initial and boundary value problem (3.3.23)–(3.3.25), then it holds:

$$\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 \leq \rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx}|_H^2 + \frac{t}{\rho} \int_0^t |\varphi_{xx}(\tau)|_H^2 d\tau \text{ for any } t \in [0, T].$$

We shall prove this inequality in the similar way as given in Step 2. By multiplying both sides of (3.3.15) by $\frac{u_{i xxxx}^n - u_{(i-1) xxxx}^n}{h} = v_{i xxxx}^n$, for any $i = 2, 3, \dots, n$ and integrating it over $(0, 1)$, we have

$$\begin{aligned} & \frac{\rho}{h} \int_0^1 (v_i^n - v_{i-1}^n) \cdot v_{i xxxx}^n dx + \frac{\gamma}{h} \int_0^1 u_{i xxxx}^n \cdot (u_{i xxxx}^n - u_{(i-1) xxxx}^n) dx \\ &= \frac{1}{h} \int_0^1 \varphi_i^n \cdot (u_{i xxxx}^n - u_{(i-1) xxxx}^n) dx \text{ for any } i = 2, 3, \dots, n. \end{aligned} \quad (3.3.37)$$

We apply the integration by parts twice to the first term in the left hand side of (3.3.37), and (3.3.16), we have

$$\begin{aligned} \frac{\rho}{h} \int_0^1 (v_i^n - v_{i-1}^n) \cdot v_{ixxxx}^n dx &= \frac{\rho}{h} \left(|v_{ixx}^n|_H^2 - \int_0^1 v_{(i-1)xx}^n \cdot v_{ixx}^n dx \right) \\ &\geq \frac{\rho}{2h} \left(|v_{ixx}^n|_H^2 - |v_{(i-1)xx}^n|_H^2 \right) \text{ for any } i = 2, 3, \dots, n. \end{aligned}$$

On the second term in the left hand side of (3.3.37), we have

$$\begin{aligned} \frac{\gamma}{h} \int_0^1 u_{ixxxx}^n \cdot (u_{ixxxx}^n - u_{(i-1)xxxx}^n) dx &= \frac{\gamma}{h} \left(|u_{ixxxx}^n|_H^2 - \int_0^1 u_{ixxxx}^n \cdot u_{(i-1)xxxx}^n dx \right) \\ &\geq \frac{\gamma}{h} \left(|u_{ixxxx}^n|_H^2 - |u_{(i-1)xxxx}^n|_H^2 \right) \text{ for any } i = 2, 3, \dots, n. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{1}{h} \int_0^1 \varphi_i^n \cdot (u_{ixxxx}^n - u_{(i-1)xxxx}^n) dx &= \int_0^1 \varphi_{ixx}^n \cdot \frac{u_{ixx}^n - u_{(i-1)xx}^n}{h} dx \\ &\leq |\varphi_{ixx}^n|_H |v_{ixx}^n|_H \text{ for any } i = 2, 3, \dots, n. \end{aligned}$$

By adding these inequalities, we see that

$$\begin{aligned} &\frac{\rho}{2h} \left(|v_{ixx}^n|_H^2 - |v_{(i-1)xx}^n|_H^2 \right) + \frac{\gamma}{h} \left(|u_{ixxxx}^n|_H^2 - |u_{(i-1)xxxx}^n|_H^2 \right) \\ &\leq |\varphi_{ixx}^n|_H |v_{ixx}^n|_H \text{ for any } i = 2, 3, \dots, n. \end{aligned} \quad (3.3.38)$$

Taking summation of both sides in (3.3.38) with respect to $i = 2, 3, \dots, k$ for any $k = 2, 3, \dots, n$, gives

$$\begin{aligned} &\frac{\rho}{2h} |v_{kxx}^n|_H^2 + \frac{\gamma}{2h} |u_{kxxxx}^n|_H^2 \\ &\leq \frac{1}{2h} (\rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx} + h v_{0xxxx}|_H^2) + \sum_{i=2}^k |\varphi_{ixx}^n|_H |v_{ixx}^n|_H^2 \text{ for any } k = 2, 3, \dots, n. \end{aligned} \quad (3.3.39)$$

Furthermore, we take summation of both sides in (3.3.39) with respect to $k = 2, 3, \dots, j$ for any $j = 2, 3, \dots, n$, again, and have

$$\begin{aligned} &\frac{1}{2h} \sum_{k=2}^j (\rho |v_{kxx}^n|_H^2 + \gamma |u_{kxxxx}^n|_H^2) \\ &\leq \frac{j-1}{2h} (\rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx} + h v_{0xxxx}|_H^2) + \sum_{k=2}^j \sum_{i=2}^k |\varphi_{ixx}^n|_H |v_{ixx}^n|_H^2 \\ &\leq \frac{j-1}{2h} (\rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx} + h v_{0xxxx}|_H^2) + \sum_{k=2}^j \sum_{i=2}^j \left(\frac{\rho}{4h(j-1)} |v_{ixx}^n|_H^2 + \frac{h(j-1)}{\rho} |\varphi_{ixx}^n|_H^2 \right) \\ &= \frac{j-1}{2h} (\rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx} + h v_{0xxxx}|_H^2) + \frac{\rho}{4h} \sum_{i=2}^j |v_{ixx}^n|_H^2 + \frac{h(j-1)^2}{\rho} \sum_{i=2}^j |\varphi_{ixx}^n|_H^2, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{h} \sum_{k=2}^j \left(\frac{\rho}{2} |v_{kxx}^n|_H^2 + \frac{\gamma}{2} |u_{kxxxx}^n|_H^2 \right) &\leq \frac{j-1}{h} \left(\frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 \right) + \frac{\rho}{4h} \sum_{k=2}^j |v_{kxx}^n|_H^2 \\ &\quad + \frac{h(j-1)^2}{\rho} \sum_{k=2}^j |\varphi_{kxx}^n|_H^2 \text{ for any } j = 2, 3, \dots, n. \end{aligned} \quad (3.3.40)$$

Here, from (3.3.39) and (3.3.40) it follows

$$\begin{aligned} &\frac{\rho}{2} |v_{kxx}^n|_H^2 + \frac{\gamma}{2} |u_{kxxxx}^n|_H^2 \\ &\leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 + h \sum_{i=2}^k |\varphi_{ixx}^n|_H |v_{ixx}^n|_H^2 \\ &\leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 + h \sum_{i=2}^k \left(\frac{(k-1)h}{\rho} |\varphi_{ixx}^n|_H^2 + \frac{\rho}{4(k-1)h} |v_{ixx}^n|_H^2 \right) \\ &= \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 + \frac{(k-1)h^2}{\rho} \sum_{i=2}^k |\varphi_{ixx}^n|_H^2 + \frac{\rho}{4(k-1)} \sum_{i=2}^k |v_{ixx}^n|_H^2 \\ &\leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 + \frac{(k-1)h^2}{\rho} \sum_{i=2}^k |\varphi_{ixx}^n|_H^2 \\ &\quad + \frac{1}{(k-1)} \left\{ (k-1) \left(\frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 \right) + \frac{h^2(k-1)^2}{\rho} \sum_{i=2}^k |\varphi_{ixx}^n|_H^2 \right\} \\ &\leq \rho |v_{0xx}|_H^2 + \gamma |u_{0xxxx} + hv_{0xxxx}|_H^2 + \frac{2kh}{\rho} \left(h \sum_{i=2}^k |\varphi_{ixx}^n|_H^2 \right) \text{ for any } k = 2, 3, \dots, n. \end{aligned} \quad (3.3.41)$$

Here, from the definition of φ_i^n (see Step 2), we see that

$$h \sum_{i=2}^k |\varphi_{ixx}^n|_H^2 \leq \int_0^{kh} |\varphi_{nxx}(t)|_H^2 dt. \quad (3.3.42)$$

Here, thanks to (3.3.41) and (3.3.42), we have

$$\begin{aligned} &\frac{\rho}{2} |v_{kxx}^n|_H^2 + \frac{\gamma}{2} |u_{kxxxx}^n|_H^2 \\ &\leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 + \frac{2kh}{\rho} \int_0^{kh} |\varphi_{nxx}(t)|_H^2 dt \text{ for any } k = 2, 3, \dots, n, \end{aligned}$$

namely,

$$\begin{aligned} \frac{\rho}{2} |v_{nxx}(kh)|_H^2 + \frac{\gamma}{2} |u_{nxxxx}(kh)|_H^2 &\leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx} + hv_{0xxxx}|_H^2 \\ &\quad + \frac{2T}{\rho} \int_0^{kh} |\varphi_{nxx}(t)|_H^2 dt \text{ for any } k = 2, 3, \dots, n. \end{aligned} \quad (3.3.43)$$

By (3.3.43), the weakly convergences of $\{v_{nxx}(t)\}$ and $\{u_{nxxxx}(t)\}$ in H for any $t \in [0, T]$ and $\{\varphi_n\} \subset L^2(0, T; V_1)$, we can easily prove the following inequality for any $t \in \mathbb{Q} \cap [0, T]$:

$$\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 \leq \rho|v_{0xx}|_H^2 + \gamma|u_{0xxxx}|_H^2 + \frac{2T}{\rho} \int_0^t |\varphi_{xx}(\tau)|_H^2 d\tau. \quad (3.3.44)$$

Let $t \in [0, T]$ and put $r = \frac{t}{T} \in [0, 1]$. Here we can take a sequence $\{r_i\} \subset \mathbb{Q} \cap [0, 1]$ such that $\{r_i\}$ is monotone increasing or monotone decreasing and $r_i \rightarrow r$ as $i \rightarrow \infty$. Clearly, by putting $t_i = r_i T$ for any $i = 1, 2, \dots$, we have $t_i \rightarrow t$ as $i \rightarrow \infty$.

By (3.3.44) we have

$$\begin{aligned} \frac{\rho}{2}|u_{txx}(t_i)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t_i)|_H^2 &\leq \rho|v_{0xx}|_H^2 + \gamma|u_{0xxxx}|_H^2 \\ &+ \frac{2T}{\rho} \int_0^{t_i} |\varphi_{xx}(\tau)|_H^2 d\tau \text{ for any } i = 1, 2, \dots \end{aligned} \quad (3.3.45)$$

Since $\varphi \in C([0, T]; V_1)$, $\{u_{txx}(t_i)\}$ and $\{u_{xxxx}(t_i)\}$ are bounded in H , $u, u_t \in C(\overline{Q(T)})$ and $t_i \rightarrow t$ as $i \rightarrow \infty$. We see that there exists a subsequence $\{t_{ij}\} \subset \{t_i\}$ such that

$$u_{txx}(t_{ij}) \rightarrow u_{txx}(t) \text{ and } u_{xxxx}(t_{ij}) \rightarrow u_{xxxx}(t) \text{ weakly in } H \text{ as } j \rightarrow \infty.$$

We note that $\lim_{i \rightarrow \infty} \int_0^{t_i} |\varphi_{xx}(\tau)|_H^2 d\tau = \int_0^t |\varphi_{xx}(\tau)|_H^2 d\tau$. By these convergences and (3.3.45), we obtain

$$\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 \leq \rho|v_{0xx}|_H^2 + \gamma|u_{0xxxx}|_H^2 + \frac{2T}{\rho} \int_0^t |\varphi_{xx}(\tau)|_H^2 d\tau \text{ for any } t \in [0, T].$$

This is the conclusion of Step 5.

Hence, from Step 4 and Step 5, Lemma 3.2 (2) can be proved for $\varphi \in C([0, T]; V_1)$ and $v_0 \in V_2$.

Finally, we prove Lemma 3.2 (2) in the case of $\varphi \in L^2(0, T; V_1)$ and $v_0 \in V_1$. Obviously, there exist $\{v_{0n}\} \subset V_2$, $\{\varphi_n\} \subset C([0, T]; V_1)$ such that

$$v_{0n} \rightarrow v_0 \text{ in } V_1, \varphi_n \rightarrow \varphi \text{ in } L^2(0, T; V_1) \text{ as } n \rightarrow \infty.$$

Subsequently, for any $n \in \mathbb{Z}_{>0}$, $u_0 \in V_2$, $v_{0n} \in V_2$ and $\varphi_n \in C([0, T]; V_1)$, Step 4 and Step 5 guarantee the existence of only one $u_n \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; H^2(0, 1)^2) \cap L^\infty(0, T; H)$ such that

$$\begin{aligned} \rho u_{ntt} + \gamma u_{nxxxx} &= \varphi_n \text{ a.e. on } Q(T), \\ \frac{\partial^i}{\partial x^i} u_n(0) &= \frac{\partial^i}{\partial x^i} u_n(1) \text{ a.e. on } (0, T) \text{ for } i = 0, 1, 2, 3, \\ u(0) &= u_0, u_t(0) = v_{0n} \text{ a.e. on } (0, 1), \end{aligned}$$

and

$$\frac{\rho}{2}|u_{ntxx}(t)|_H^2 + \frac{\gamma}{2}|u_{nxxxx}(t)|_H^2 \leq \rho|v_{0nxx}|_H^2 + \gamma|u_{0xxxx}|_H^2 + \frac{2T}{\rho} \int_0^t |\varphi_{nxx}(\tau)|_H^2 d\tau \text{ for any } t \in [0, T].$$

Similarly to those of Step 4 and Step 5, we can prove Lemma 3.2 (2) for $\varphi \in L^2(0, T; V_1)$ and $v_0 \in V_1$. Hence, Lemma 3.2 has been proved, completely. \square

Proof of Lemma 3.3. Let $u_{n0} \in V_2$, $v_{n0} \in V_1$, $\widehat{F}_n \in C_0^\infty(Q(T))$, $\widehat{\varphi}_n \in L^2(0, T; V_1)$, and $w_n \in L^2(0, T; H^4(0, 1)^2)$. We can easily show $\left(\widehat{F}_n w_{nx}\right)_x \in L^2(0, T; V_1)$. Accordingly, by Lemma 3.2 (2), there exists a solution $\widehat{\eta}_n \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; H^2(0, 1)^2) \cap L^\infty(0, T; H^4(0, 1)^2)$ satisfies (3.3.7)–(3.3.9). We define an operator $\Lambda : L^2(0, T; H^4(0, 1)^2) \rightarrow L^2(0, T; H^4(0, 1)^2)$ by $\widehat{\eta}_n = \Lambda w_n$.

First, we show the uniqueness of the solution w_n . Let $\widehat{\eta}_{n1}$, $\widehat{\eta}_{n2}$ be solutions of (3.3.7)–(3.3.9) corresponding to $w_{n1}, w_{n2} \in L^2(0, T; H^4(0, 1)^2)$, respectively, and put $\widehat{\eta}_n = \widehat{\eta}_{n1} - \widehat{\eta}_{n2}$ and $w_n = w_{n1} - w_{n2}$. It is clear that $\widehat{\eta}_n$ satisfies

$$\rho \widehat{\eta}_{ntt} + \gamma \widehat{\eta}_{nxxxx} - \left(\widehat{F}_n w_{nx}\right)_x = 0 \text{ a.e. on } Q(T), \quad (3.3.46)$$

$$\frac{\partial^i}{\partial x^i} \widehat{\eta}_n(0) = \frac{\partial^i}{\partial x^i} \widehat{\eta}_n(1) \text{ a.e. on } (0, T) \text{ and } i = 0, 1, 2, 3, \quad (3.3.47)$$

$$\widehat{\eta}_n(0) = 0, \widehat{\eta}_{nt}(0) = 0 \text{ a.e. on } (0, 1). \quad (3.3.48)$$

Multiplying both sides (3.3.46) by $\widehat{\eta}_{nt}$ and integrating on $(0, 1)$, we have

$$\rho \int_0^1 \widehat{\eta}_{ntt} \cdot \widehat{\eta}_{nt} dx + \gamma \int_0^1 \widehat{\eta}_{nxxxx} \cdot \widehat{\eta}_{nt} dx = \int_0^1 \left(\widehat{F}_n w_{nx}\right)_x \cdot \widehat{\eta}_{nt} dx \text{ a.e. on } (0, T). \quad (3.3.49)$$

By integrating by parts (3.3.49) with respect to x and (3.3.47), we see that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\rho}{2} |\widehat{\eta}_{nt}|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxx}|_H^2 \right) &\leq \frac{\rho}{2} |\widehat{\eta}_{nt}|_H^2 + \frac{1}{2\rho} |(\widehat{F}_n w_{nx})_x|_H^2 \\ &\leq \left(\frac{\rho}{2} |\widehat{\eta}_{nt}|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxx}|_H^2 \right) + \frac{1}{2\rho} |(\widehat{F}_n w_{nx})_x|_H^2 \text{ a.e. on } (0, T). \end{aligned}$$

Here, by applying the Gronwall inequality with the initial condition, we obtain

$$\frac{\rho}{2} |\widehat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxx}(t)|_H^2 \leq \frac{e^t}{2\rho} \int_0^t |(\widehat{F}_n w_{nx})_x(\tau)|_H^2 d\tau \text{ for any } t \in [0, T].$$

Since $\widehat{F}_n \in C_0^\infty(Q(T))$, by taking a positive constant $C_1 = \max_{(t,x) \in Q(T)} \{|\widehat{F}_{nx}(t, x)|^2 + |\widehat{F}_n(t, x)|^2\}$, we have

$$\begin{aligned} \frac{e^t}{2\rho} \int_0^t |(\widehat{F}_n w_{nx})_x(\tau)|_H^2 d\tau &\leq \frac{e^t}{\rho} \left(\int_0^t |(\widehat{F}_{nx} w_{nx})(\tau)|_H^2 d\tau + \int_0^t |(\widehat{F}_n w_{nxx})(\tau)|_H^2 d\tau \right) \\ &\leq \frac{2C_1 e^t}{\rho} \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T]. \end{aligned}$$

Therefore, we obtain

$$\frac{\rho}{2} |\widehat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxx}(t)|_H^2 \leq \frac{2C_1 e^t}{\rho} \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T]. \quad (3.3.50)$$

Moreover, by $\left(\widehat{F}_n w_{nx}\right)_x \in L^2(0, T; V_1)$ and the inequality of Lemma 3.2 (2), we have

$$\frac{\rho}{2} |\widehat{\eta}_{ntxx}(t)|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxxxx}(t)|_H^2 \leq \frac{2T}{\rho} \int_0^t \left| \left(\widehat{F}_n w_{nx}\right)_{xxx}(\tau) \right|_H^2 d\tau \text{ for any } t \in [0, T].$$

Since $\left(\widehat{F}_n w_{nx}\right)_{xxx} = \widehat{F}_{nxxx} w_{nx} + 3\widehat{F}_{nxx} w_{nxx} + 3\widehat{F}_{nx} w_{nxxx} + \widehat{F}_n w_{nxxxx}$ on $Q(T)$ and $\widehat{F}_n \in C_0^\infty(Q(T))$, there exists a positive constant C_2 such that

$$\left| \frac{\partial^i \widehat{F}_n}{\partial x^i} \right| \leq C_2 \text{ on } Q(T) \text{ for any } i = 0, 1, 2, 3.$$

Accordingly, we have

$$\frac{2T}{\rho} \int_0^t \left| \left(\widehat{F}_n w_{nx}\right)_{xxx}(\tau) \right|_H^2 d\tau \leq C_3 \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T],$$

where $C_3 = \frac{288C_2^2 T}{\rho}$. Thus, we obtain

$$\frac{\rho}{2} |\widehat{\eta}_{ntxx}(t)|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxxxx}(t)|_H^2 \leq C_3 \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T]. \quad (3.3.51)$$

By (3.3.50) and (3.3.51), we obtain

$$|\widehat{\eta}_{nt}(t)|_H^2 + |\widehat{\eta}_{nxx}(t)|_H^2 + |\widehat{\eta}_{ntxx}(t)|_H^2 + |\widehat{\eta}_{nxxxx}(t)|_H^2 \leq C_4 \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \quad (3.3.52)$$

for any $t \in [0, T]$, where $C_4 = \frac{2}{\min\{\rho, \gamma\}} \left\{ \frac{2C_1 e^T}{\rho} + C_3 \right\}$.

Now, by using (3.3.48) and (3.3.52), we have

$$\begin{aligned} |\widehat{\eta}_n(t)|_H^2 &= \int_0^1 \left| \int_0^t \widehat{\eta}_{n\tau}(\tau, x) d\tau \right|^2 dx \\ &\leq t \int_0^t |\widehat{\eta}_{n\tau}(\tau)|_H^2 d\tau \\ &\leq C_4 t^2 \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T]. \end{aligned} \quad (3.3.53)$$

By integrating by parts and using (3.3.47), (3.3.52) and (3.3.53), we obtain

$$\begin{aligned} |\widehat{\eta}_{nx}(t)|_H^2 &\leq |\widehat{\eta}_n(t)|_H |\widehat{\eta}_{nxx}(t)|_H \\ &\leq C_4 t \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T], \end{aligned} \quad (3.3.54)$$

$$\begin{aligned} |\widehat{\eta}_{nxxx}(t)|_H^2 &\leq |\widehat{\eta}_{nxx}(t)|_H |\widehat{\eta}_{nxxxx}(t)|_H \\ &\leq C_4 \int_0^t |w_n(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T], \end{aligned} \quad (3.3.55)$$

By using (3.3.52)–(3.3.55), and recalling $\widehat{\eta}_n = \widehat{\eta}_{n1} - \widehat{\eta}_{n2}$ and $w_n = w_{n1} - w_{n2}$, we infer that

$$\int_0^t |(\Lambda w_{n1})(\tau) - (\Lambda w_{n2})(\tau)|_{H^4(0,1)^2}^2 d\tau \leq C_5 t \int_0^t |w_{n1}(\tau) - w_{n2}(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T],$$

where $C_5 = C_4(T^2 + T + 3)$. Now, the following assertion holds:

$$\begin{aligned} & \int_0^t |(\Lambda^m w_{n1})(\tau) - \Lambda^m w_{n2}(\tau)|_{H^4(0,1)^2}^2 d\tau \\ & \leq \frac{(C_5 t)^m}{m!} \int_0^t |w_{n1}(\tau) - w_{n2}(\tau)|_{H^4(0,1)^2}^2 d\tau \text{ for any } t \in [0, T] \text{ and } m \in \mathbb{Z}_{>0}. \end{aligned}$$

This assertion can be proved by the mathematical induction. It is clear that for $M \in \mathbb{Z}_{>0}$ with $\frac{(C_5 t)^M}{M!} < 1$, Λ^M is a contraction mapping on $L^2(0, T; H^4(0, 1)^2)$. Thus, by the Banach fixed point theorem, there exists $\hat{\eta}_n \in L^2(0, T; H^4(0, 1)^2)$ only one such that $\Lambda^M \hat{\eta}_n = \hat{\eta}_n$. Therefore, $\Lambda^M(\Lambda \hat{\eta}_n) = \Lambda \hat{\eta}_n$, namely, $\Lambda \hat{\eta}_n$ is also the fixed point of Λ^M . By the uniqueness of the fixed point of Λ^M , we see that $\Lambda \hat{\eta}_n = \hat{\eta}_n$. Clearly, $\hat{\eta}_n$ satisfies (3.3.7)–(3.3.9). Hence, Lemma 3.3 has been proved. \square

Proof of Lemma 3.1. First, we put $\hat{\eta}_n(t) = \eta_n(T - t)$, $\hat{F}_n(t) = F_n(T - t)$, $\hat{\varphi}_n(t) = \varphi_n(T - t)$ for any $t \in [0, T]$ in (3.3.4)–(3.3.6).

$$\rho \hat{\eta}_{ntt} + \gamma \hat{\eta}_{nxxxx} - \left(\hat{F}_n \hat{\eta}_{nx} \right)_x = \hat{\varphi} \text{ in } Q(T), \quad (3.3.56)$$

$$\frac{\partial^i}{\partial x^i} \hat{\eta}_n(t, 0) = \frac{\partial^i}{\partial x^i} \hat{\eta}_n(t, 1) \text{ for } t \in [0, T] \text{ and } i = 0, 1, 2, 3, \quad (3.3.57)$$

$$\hat{\eta}_n(0) = \hat{\eta}_{nt}(0) = 0 \text{ on } (0, 1). \quad (3.3.58)$$

The existence of a solution to the above problem is a direct consequence of Lemma 3.3. Hence, to accomplish the proof of Lemma 3.1, it is sufficient to show uniqueness of the solution.

Let $\hat{\eta}_{n1}, \hat{\eta}_{n2}$ be solutions of (3.3.56)–(3.3.58) and $\hat{\eta}_n = \hat{\eta}_{n1} - \hat{\eta}_{n2}$. Easily, we have

$$\rho \hat{\eta}_{ntt} + \gamma \hat{\eta}_{nxxxx} = \left(\hat{F}_n \hat{\eta}_{nx} \right)_x \text{ a.e. on } Q(T). \quad (3.3.59)$$

Multiplying both sides of (3.3.59) by $\hat{\eta}_{nt}$ and integrating it on $(0, 1)$, we see that

$$\rho \int_0^1 \hat{\eta}_{ntt} \cdot \hat{\eta}_{nt} dx + \gamma \int_0^1 \hat{\eta}_{nxxxx} \cdot \hat{\eta}_{nt} dx = \int_0^1 \left(\hat{F}_n \hat{\eta}_{nx} \right)_x \cdot \hat{\eta}_{nt} dx \text{ a.e. on } (0, T).$$

By elementary calculations, for some $C_1, C_2 > 0$, it holds that

$$\frac{d}{dt} \left(\frac{\rho}{2} |\hat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\hat{\eta}_{nxx}(t)|_H^2 \right) \leq C_1 \left(\frac{\rho}{2} |\hat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\hat{\eta}_{nxx}(t)|_H^2 \right) + C_2 |\hat{\eta}_{nx}(t)|_H^2 \text{ for a.e. } t \in (0, T). \quad (3.3.60)$$

Here, by using (3.3.57) and (3.3.58), the following inequalities are obtained:

$$|\hat{\eta}_{nx}(t)|_H^2 \leq \frac{1}{2} (|\hat{\eta}_n(t)|_H^2 + |\hat{\eta}_{nxx}(t)|_H^2), |\hat{\eta}_n(t)|_H^2 = \int_0^1 \left| \int_0^t \hat{\eta}_{n\tau}(\tau) d\tau \right|^2 dx \leq t \int_0^t |\hat{\eta}_{n\tau}(\tau)|_H^2 d\tau$$

for any $t \in [0, T]$.

By adding these inequalities and (3.3.60), we have

$$\frac{d}{dt} \left(\frac{\rho}{2} |\hat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\hat{\eta}_{nxx}(t)|_H^2 \right) \leq C'_1 \left(\frac{\rho}{2} |\hat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\hat{\eta}_{nxx}(t)|_H^2 \right) + \frac{C'_2}{2} t \int_0^t |\hat{\eta}_{n\tau}(\tau)|_H^2 d\tau$$

for a.e. $t \in (0, T)$, where C'_1, C'_2 are positive constants. Thanks to the Gronwall inequality, we get

$$\begin{aligned} \frac{\rho}{2} |\widehat{\eta}_{nt}(t)|_H^2 + \frac{\gamma}{2} |\widehat{\eta}_{nxx}(t)|_H^2 &\leq \frac{e^{C'_1 t} C'^2_2}{2} \int_0^t t \int_0^t |\widehat{\eta}_{nt}(\xi)|_H^2 d\xi d\tau \\ &\leq \frac{e^{C'_1 T} C'^2_2}{2} t^3 \sup_{0 \leq \xi \leq t} |\widehat{\eta}_{nt}(\xi)|_H^2 \text{ for any } t \in [0, T]. \end{aligned}$$

Hence, it holds that

$$\frac{\rho}{2} \sup_{0 \leq t \leq T_1} |\widehat{\eta}_{nt}(t)|_H^2 \leq C_3 T_1^3 \sup_{0 \leq t \leq T_1} |\widehat{\eta}_{nt}(t)|_H^2$$

where $C_3 = \frac{e^{C'_1 T} C'^2_2}{2}$. Here, by choosing $T_1 > 0$ such that $C_3 T_1^3 \leq \frac{\rho}{4}$, we conclude that $|\widehat{\eta}_{nt}(t)|_H = 0$ for any $t \in [0, T_1]$. Recursively, we can prove the uniqueness of (3.3.56)–(3.3.58). Hence, Lemma 3.1 has been proved. \square

The following Lemmas 3.5 and 3.7 are keys in the proof of the uniqueness of solutions to P_0 . In the following argument, we use the notation η_n as the solution of (3.3.56)–(3.3.58).

Lemma 3.5. *For each $t \in [0, T]$ there exists a unique strong solution $\xi_n(t) \in V_1$ of the boundary value problem (3.3.61) and (3.3.62):*

$$-\xi_{nxx}(t) + \xi_n(t) = \eta_n(t) \text{ on } (0, 1), \quad (3.3.61)$$

$$\xi_n(t, 0) = \xi_n(t, 1) \text{ and } \xi_{nx}(t, 0) = \xi_{nx}(t, 1). \quad (3.3.62)$$

Moreover, it holds that $\xi_n \in W^{2,2}(0, T; H^2(0, 1)^2)$ and

$$-\xi_{nttxx}(t) + \xi_{ntt}(t) = \eta_{ntt}(t) \text{ on } (0, 1), \quad (3.3.63)$$

$$\xi_{ntt}(t, 0) = \xi_{ntt}(t, 1), \xi_{nttx}(t, 0) = \xi_{nttx}(t, 1) \text{ for a.e. } t \in [0, T]. \quad (3.3.64)$$

The first assertion of this lemma is proved by the Riesz representation theorem to the Hilbert space $X := \{z \in H^1(0, 1)^2 | z(0) = z(1)\}$ with the standard inner product for $H^1(0, 1)^2$. In the proof to the differentiability of ξ_n with respect to time, we use the following lemma.

Lemma 3.6. *Let $T > 0$. If $f \in W^{1,2}(0, T; H)$, then the following convergence holds:*

$$\left| \frac{f(t+h) - f(t)}{h} - f_t(t) \right|_H \rightarrow 0 \text{ as } h \rightarrow 0 \text{ for a.e. } t \in [0, T].$$

Proof of Lemma 3.6. First, we note that $H = L^2(0, 1)^2$ is separable. Hence, we can take a sequence $\{\zeta_i\}_{i=1}^\infty \subset H$ satisfying for any $g \in H$ and $\varepsilon > 0$ there exists $i_0 \in \mathbb{Z}_{>0}$ such that $|g - \zeta_{i_0}|_H < \varepsilon$. Put $g_i(t) = |f_t(t) - \zeta_i|_H^2$ for any $t \in [0, T]$ and $i \in \mathbb{Z}_{>0}$. Clearly, $f_t \in L^2(0, T; H)$ implies $g_i \in L^1(0, T)^2$ for any $i \in \mathbb{Z}_{>0}$. Thus, by using the Lebesgue differentiation theorem ([28, Theorem 7.7]), the following convergence holds.

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} |g_i(\tau) - g_i(t)| d\tau = 0 \text{ for a.e. } t \in [0, T],$$

namely,

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} g_i(\tau) d\tau = g_i(t) \quad \text{for a.e. } t \in [0, T]. \quad (3.3.65)$$

This means that for any $i \in \mathbb{Z}_{>0}$, there exists a set $E_i \subset [0, T]$ having null measure such that $\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} g_i(\tau) d\tau = g_i(t)$ for $t \notin E_i$. It is obvious that the measure of $E = \bigcup_{i=1}^{\infty} E_i$ is zero. Here, for $t \notin E$ and $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{Z}_{>0}$ such that $|f_t(t) - \zeta_{i_\varepsilon}|_H < \frac{\varepsilon}{3}$. By the above convergence, we observe that for $t \notin E$,

$$\begin{aligned} \left| \frac{f(t+h) - f(t)}{h} - f_t(t) \right|_H &\leq \left| \frac{1}{h} \int_t^{t+h} (f_t(\tau) - f_t(t))^2 d\tau \right|_H \\ &= \left| \frac{1}{h} \int_0^1 \int_t^{t+h} |f_t(\tau, x) - f_t(t, x)|^2 d\tau dx \right|^{\frac{1}{2}} \\ &= \left| \frac{1}{h} \int_t^{t+h} |f_t(\tau) - f_t(t)|_H^2 d\tau \right|^{\frac{1}{2}} \\ &\leq \left| \frac{1}{h} \int_t^{t+h} (|f_t(\tau) - \zeta_i|_H + |\zeta_i - f_t(t)|_H)^2 d\tau \right|^{\frac{1}{2}} \\ &\leq \left| \frac{1}{h} \int_t^{t+h} |f_t(\tau) - \zeta_i|_H^2 d\tau \right|^{\frac{1}{2}} + \left| \frac{1}{h} \int_t^{t+h} |\zeta_i - f_t(t)|_H^2 d\tau \right|^{\frac{1}{2}} \\ &\leq \left| \frac{1}{h} \int_t^{t+h} |f_t(\tau) - \zeta_i|_H^2 d\tau \right|^{\frac{1}{2}} + \frac{\varepsilon}{3}. \end{aligned}$$

From (3.3.65), there exists $\delta > 0$ such that if $|h| < \delta$, then $\left| \frac{1}{h} \int_t^{t+h} g_i(\tau) d\tau - g_i(t) \right| < \left(\frac{\varepsilon}{3} \right)^2$, namely, $\left| \frac{1}{h} \int_t^{t+h} |f_t(\tau) - \zeta_i|_H^2 d\tau - |f_t(t) - \zeta_i|_H^2 \right| < \left(\frac{\varepsilon}{3} \right)^2$. Hence, we obtain

$$\left| \frac{f(t+h) - f(t)}{h} - f_t(t) \right|_H < \varepsilon \quad \text{for } |h| < \delta \text{ and } t \notin E.$$

Thus, Lemma 3.6 has been proved. \square

Before proving Lemma 3.5, we define a weak solution and a strong solution of the boundary value problem, (3.3.61)–(3.3.62).

Definition 3.2. For given $\eta_n(t)$ for $t \in [0, T]$, a function $\xi_n(t)$ on $(0, 1)$ is called a weak solution of (3.3.61)–(3.3.62) on $(0, 1)$ if $\xi_n(t)$ satisfies that $\xi_n(t) \in X$ and

$$\int_0^1 \xi_{nx}(t, x) \cdot z_x(x) dx + \int_0^1 \xi_n(t, x) \cdot z(x) dx = \int_0^1 \eta_n(t, x) \cdot z(x) dx \quad \text{for any } z \in X.$$

We note that the left hand side of the above weak formulation gives an inner product of X .

Definition 3.3. A function $\xi_n(t)$ on $(0, 1)$ is called a strong solution of (3.3.61)–(3.3.62) on $(0, 1)$ if ξ_n satisfies that $\xi_n(t) \in V_1$ and satisfies (3.3.61) and (3.3.62) in a usual sense.

Proof of Lemma 3.5. For $t \in [0, T]$ and $n \in \mathbb{Z}_{>0}$. We define a functional L by

$$\langle L, z \rangle = \int_0^1 \eta_n(t, x) \cdot z(x) dx, \text{ for any } z \in X.$$

Since $\eta_n(t) \in H$, L is the element of the dual space of X . Hence, by using the Riesz representation theorem, there exists only one $\xi_n(t) \in X$ satisfying $(\xi_n(t), z)_X = \int_0^1 \eta_n(t, x) \cdot z(x) dx$ for any $z \in X$. Thus, existence and uniqueness of a weak solution to (3.3.61)–(3.3.62) hold.

Next, we show that the weak solution of (3.3.61)–(3.3.62) is its strong solution. Let $z \in C_0^\infty(0, 1)^2$. We have

$$\begin{aligned} \langle \xi_{nxx}(t), z \rangle_{C_0^\infty(0, 1)^2} &= - \int_0^1 \xi_{nx}(t, x) \cdot z_x(x) dx \\ &= \int_0^1 (-\eta_n(t, x) + \xi_n + \xi_n(t, x)) \cdot z(x) dx \quad \text{for any } z \in C_0^\infty(0, 1)^2, \end{aligned}$$

where the notation $\langle, \rangle_{C_0^\infty(0, 1)^2}$ represents the duality pair in $C_0^\infty(0, 1)^2$. From the definition of derivative in the sense of distributions, we have $\xi_{nxx}(t) \in L^2(0, 1)^2$ and

$$\xi_{nxx}(t) = -\eta_n(t) + \xi_n(t) \text{ a.e. on } (0, 1) \text{ for any } t \in [0, T] \text{ and } n \in \mathbb{Z}_{>0}. \quad (3.3.66)$$

Thus, $\xi_n(t) \in H^2(0, 1)^2$ for any $t \in [0, T]$. The definition of X implies $\xi_n(t, 0) = \xi_n(t, 1)$ for any $t \in [0, T]$. Moreover, $\xi_{nx}(t, 0) = \xi_{nx}(t, 1)$ holds for any $t \in [0, T]$. Indeed, for any $r \in \mathbb{R}$ and $z \in C_0^\infty(0, 1)^2$ we put $\widehat{z} = z + r$. Thanks to $\widehat{z} \in X$ and the weak formulation in Definition 3.2, we have

$$\begin{aligned} - \int_0^1 \xi_{nxx}(t, x) \cdot \widehat{z}(x) dx &= \widehat{z}(0) \cdot (-\xi_{nx}(t, 1) + \xi_{nx}(t, 0)) + \int_0^1 \xi_{nx}(t, x) \cdot \widehat{z}_x(x) dx \\ &= r(-\xi_{nx}(t, 1) + \xi_{nx}(t, 0)) + \int_0^1 \xi_{nx}(t, x) \cdot \widehat{z}_x(x) dx \\ &= r(-\xi_{nx}(t, 1) + \xi_{nx}(t, 0)) - \int_0^1 (\eta_n(t, x) - \xi_n(t, x)) \cdot \widehat{z}(x) dx \end{aligned}$$

for any $t \in [0, T]$ and $n \in \mathbb{Z}_{>0}$. By (3.3.66), we obtain

$$r(-\xi_{nx}(t, 1) + \xi_{nx}(t, 0)) = 0 \quad \text{for any } t \in [0, T], n \in \mathbb{Z}_{>0} \text{ and } r \in \mathbb{R},$$

and

$$\xi_{nx}(t, 0) = \xi_{nx}(t, 1) \text{ for any } t \in [0, T] \text{ and } n \in \mathbb{Z}_{>0}.$$

Therefore, $\xi_n(t) \in V_1$ for any $t \in [0, T]$ and $n \in \mathbb{Z}_{>0}$. Thus, we have proved the first assertion of this lemma.

Finally, we prove the differentiability of ξ_n with respect to time. By the first assertion of this lemma with $\eta_{nt}(t) \in H$ for any $n \in \mathbb{Z}_{>0}$ and $t \in [0, T]$, there exists $\widehat{\xi}_n(t) \in V_1$ satisfies

$$-\widehat{\xi}_{nxx}(t) + \widehat{\xi}_n(t) = \eta_{nt}(t) \text{ on } (0, 1), \quad (3.3.67)$$

$$\widehat{\xi}_n(t, 0) = \widehat{\xi}_n(t, 1), \widehat{\xi}_{nx}(t, 0) = \widehat{\xi}_{nx}(t, 1) \text{ for a.e. } t \in (0, T).$$

By (3.3.61) and (3.3.62), we have

$$-\left(\frac{\xi_{nxx}(t+h) - \xi_{nxx}(t)}{h} - \widehat{\xi}_{nxx}(t)\right) + \frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t) = \frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)$$

a.e. on $(0, 1)$. Multiplying both sides of the above equation by $\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)$ and integrating it on $(0, 1)$ with respect to x , we have

$$\begin{aligned} & -\int_0^1 \left(\frac{\xi_{nxx}(t+h) - \xi_{nxx}(t)}{h} - \widehat{\xi}_{nxx}(t)\right) \left(\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right) dx + \left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_H^2 \\ & = \int_0^1 \left(\frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)\right) \left(\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right) dx \quad \text{for any } t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} & \left|\frac{\xi_{nxx}(t+h) - \xi_{nxx}(t)}{h} - \widehat{\xi}_{nxx}(t)\right|_H^2 + \left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_H^2 \\ & = \int_0^1 \left(\frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)\right) \left(\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right) dx \\ & \leq \frac{1}{2} \left(\left|\frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)\right|_H^2 + \left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_H^2 \right), \end{aligned}$$

namely,

$$\left|\frac{\xi_{nxx}(t+h) - \xi_{nxx}(t)}{h} - \widehat{\xi}_{nxx}(t)\right|_H^2 + \frac{1}{2} \left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_H^2 \leq \frac{1}{2} \left|\frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)\right|_H^2 \quad (3.3.68)$$

for any $t \in [0, T]$. Since, $\eta_n \in W^{1,2}(0, T; H)$, by using Lemma 3.6, we see that

$$\left|\frac{\eta_n(t+h) - \eta_n(t)}{h} - \eta_{nt}(t)\right|_H \rightarrow 0 \text{ as } h \rightarrow 0 \text{ for a.e. } t \in [0, T]. \quad (3.3.69)$$

Accordingly, by (3.3.68) and (3.3.69), we have

$$\left|\frac{\xi_{nxx}(t+h) - \xi_{nxx}(t)}{h} - \widehat{\xi}_{nxx}(t)\right|_H \rightarrow 0, \left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_H \rightarrow 0 \text{ as } h \rightarrow 0,$$

namely,

$$\left|\frac{\xi_n(t+h) - \xi_n(t)}{h} - \widehat{\xi}_n(t)\right|_{H^1(0,1)^2} \rightarrow 0 \text{ as } h \rightarrow 0 \text{ for a.e. } t \in [0, T].$$

Hence, $\xi_{nt}(t) = \widehat{\xi}_n(t)$ in V_1 for a.e. $t \in [0, T]$.

For any $t \in [0, T]$, we multiply both sides of (3.3.67) by $\widehat{\xi}_n(t)$ and integrate it on $(0, 1)$ with respect to x , and have

$$-\int_0^1 \widehat{\xi}_{nxx}(t) \cdot \widehat{\xi}_n(t) dx + \int_0^1 |\widehat{\xi}_n(t)|^2 dx = \int_0^1 \eta_{nt}(t) \cdot \widehat{\xi}_n(t) dx,$$

$$\left| \widehat{\xi}_{nx}(t) \right|_H^2 + \left| \widehat{\xi}_n(t) \right|_H^2 = \int_0^1 \eta_{nt}(t) \cdot \widehat{\xi}_n(t) dx \leq \frac{1}{2} \left(\left| \eta_{nt}(t) \right|_H^2 + \left| \widehat{\xi}_n(t) \right|_H^2 \right),$$

that is,

$$\left| \widehat{\xi}_n(t) \right|_{V_1}^2 \leq \left| \eta_{nt}(t) \right|_H^2 \quad \text{for any } t \in [0, T]. \quad (3.3.70)$$

Since, $\eta_{nt} \in L^2(0, T; H)$, $\xi_{nt}(t) = \widehat{\xi}_n(t)$ in V_1 for a.e. $t \in [0, T]$ and (3.3.70), we see that $\xi_{nt} \in L^2(0, T; V_1)$ for any $n \in \mathbb{Z}_{>0}$. Here, we note that $\eta_{tt} \in L^2(0, T; H)$. Hence, by similar argument as above, we can prove $\xi_{tt} \in L^2(0, T; H)$ satisfying $\xi_{ntt}(t) = \widehat{\xi}_{nt}(t)$ in V_1 for a.e. $t \in [0, T]$ and (3.3.63)–(3.3.64). Thus, Lemma 3.5 has been proved. \square

From Lemma 3.5 we can get the following uniform estimate for η_{nx} with respect to n .

Lemma 3.7. *There exists $\alpha > 0$ such that*

$$\left| \eta_{nx} \right|_H \leq \alpha \text{ on } [0, T] \text{ for } n \in \mathbb{Z}_{>0}.$$

Proof of Lemma 3.7. For $n \in \mathbb{Z}_{>0}$, let η_n be a solution for the approximate dual problem (3.3.4) - (3.3.6), and ξ_n be a solution of (3.3.62). By putting $\widehat{\eta}_n(t) = \eta(T - t)$, $\widehat{\xi}_n(t) = \xi(T - t)$, $\widehat{F}_n(t) = F_n(T - t)$ and $\widehat{\varphi}(t) = \varphi(T - t)$ for $t \in (0, T)$, and $n \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} \rho \widehat{\eta}_{ntt} + \gamma \widehat{\eta}_{nxxxx} - \left(\widehat{F}_n \widehat{\eta}_{nx} \right)_x &= \widehat{\varphi} \text{ in } Q(T), \\ \widehat{\eta}_n(0) &= \widehat{\eta}_{nt}(0) = 0 \text{ on } (0, 1), \\ \frac{\partial^i \widehat{\eta}_n}{\partial x^i}(t, 0) &= \frac{\partial^i \widehat{\eta}_n}{\partial x^i}(t, 1) \text{ on } (0, T) \text{ for } i = 0, 1, 2, 3, \end{aligned} \quad (3.3.71)$$

and

$$\begin{aligned} -\widehat{\xi}_{nttxx}(t) + \widehat{\xi}_{ntt}(t) &= \widehat{\eta}_{ntt}(t) \text{ in } (0, 1), \\ \widehat{\xi}_{ntt}(t, 0) &= \widehat{\xi}_{ntt}(t, 1) \text{ and } \widehat{\xi}_{nttx}(t, 0) = \widehat{\xi}_{nttx}(t, 1) \text{ for a.e. } t \in (0, T). \end{aligned} \quad (3.3.72)$$

We multiply $\widehat{\xi}_{nt}$ by both sides of (3.3.71) and (3.3.72), and we have

$$\rho \widehat{\eta}_{ntt} \cdot \widehat{\xi}_{nt} + \gamma \widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt} = \left(\widehat{F}_n \widehat{\eta}_{nx} \right)_x \cdot \widehat{\xi}_{nt} + \widehat{\varphi} \cdot \widehat{\xi}_{nt} \text{ in } Q(T), \quad (3.3.73)$$

$$-\widehat{\xi}_{nttxx} \cdot \widehat{\xi}_{nt} + \widehat{\xi}_{ntt} \cdot \widehat{\xi}_{nt} = \widehat{\eta}_{ntt} \cdot \widehat{\xi}_{nt} \text{ in } Q(T). \quad (3.3.74)$$

By substituting (3.3.74) into (3.3.73), we have

$$\begin{aligned} &-\rho \int_0^1 \widehat{\xi}_{nttxx} \cdot \widehat{\xi}_{nt} dx + \rho \int_0^1 \widehat{\xi}_{ntt} \cdot \widehat{\xi}_{nt} dx + \gamma \int_0^1 \widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt} dx \\ &= \int_0^1 \left(\widehat{F}_n \widehat{\eta}_{nx} \right)_x \cdot \widehat{\xi}_{nt} dx + \int_0^1 \widehat{\varphi} \cdot \widehat{\xi}_{nt} dx \quad \text{a.e. on } [0, T]. \end{aligned}$$

Here, we note that

$$\int_0^1 \widehat{\eta}_{nxxxx} \cdot \widehat{\xi}_{nt} dx = \int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\xi}_{ntxx} dx$$

$$= - \int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\eta}_{nt} dx + \int_0^1 \widehat{\eta}_{nxx} \cdot \widehat{\xi}_{nt} dx \quad \text{on } [0, T].$$

Accordingly, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\rho \left| \widehat{\xi}_{nxt} \right|_H^2 + \rho \left| \widehat{\xi}_{nt} \right|_H^2 + \gamma \left| \widehat{\eta}_{nx} \right|_H^2 \right) \\ &= \gamma \int_0^1 \widehat{\eta}_{nx} \cdot \widehat{\xi}_{nxt} dx - \int_0^1 \widehat{F}_n \widehat{\eta}_{nx} \cdot \widehat{\xi}_{nxt} dx + \int_0^1 \widehat{\varphi} \cdot \widehat{\xi}_{nt} dx \quad \text{a.e. on } [0, T]. \end{aligned}$$

Since $\varphi \in C_0^\infty(Q(T))^2$, there exists a positive constant C_1 such that $|\varphi(t, x)| \leq C_1$ for $(t, x) \in Q(T)$, we have

$$\begin{aligned} & \frac{d}{dt} \left(\rho \left| \widehat{\xi}_{nxt} \right|_H^2 + \rho \left| \widehat{\xi}_{nt} \right|_H^2 + \gamma \left| \widehat{\eta}_{nx} \right|_H^2 \right) \\ & \leq C_2 \left\{ \rho \left| \widehat{\xi}_{nxt} \right|_H^2 + \rho \left| \widehat{\xi}_{nt} \right|_H^2 + \gamma \left| \widehat{\eta}_{nx} \right|_H^2 \right\} + C_1^2 \quad \text{a.e. on } [0, T], \end{aligned}$$

where C_2 is a positive constant depending only on ρ, γ and $\max_{n \in \mathbb{Z}_{>0}} \left| \widehat{F}_n \right|_{L^\infty(Q(T))}$. By applying Gronwall's inequality, we obtain

$$\begin{aligned} & \rho \left| \widehat{\xi}_{nxt} \right|_H^2 + \rho \left| \widehat{\xi}_{nt} \right|_H^2 + \gamma \left| \widehat{\eta}_{nx} \right|_H^2 \\ & \leq e^{C_2 T} \left(\rho \left| \widehat{\xi}_{nxt}(0) \right|_H^2 + \rho \left| \widehat{\xi}_{nt}(0) \right|_H^2 + \gamma \left| \widehat{\eta}_{nx}(0) \right|_H^2 + C_1^2 T \right) \quad \text{on } [0, T]. \end{aligned}$$

Hence, this lemma is proved. \square

Proof of the uniqueness. Let $n \in \mathbb{Z}_{>0}$ and $\varphi \in C_0^\infty(Q(T))^2$. By Lemma 3.1, there exists a solution $\eta_n \in W$. From (3.3.4), integration by parts and (3.3.2), it follows

$$\begin{aligned} \left| \int_{Q(T)} u \cdot \varphi dx dt \right| &= \left| \int_{Q(T)} u \{ \rho \eta_{ntt} + \gamma \eta_{nxxx} \} dx dt + \int_{Q(T)} u (F_n \eta_{nx})_x dx dt \right| \\ &= \left| \int_{Q(T)} (F_n - F) u_x \cdot \eta_{nx} dx dt \right| \quad \text{for each } n \in \mathbb{Z}_{>0}. \end{aligned}$$

Thanks to Lemma 3.7, we have

$$\begin{aligned} \left| \int_{Q(T)} u \cdot \varphi dx dt \right| &\leq \alpha \|u_x\|_{L^\infty(Q(T))} \int_0^T \|F_n - F\|_H dt \\ &\leq \alpha \sqrt{T} \|u_x\|_{L^\infty(Q(T))} \|F_n - F\|_{L^2(Q(T))} \quad \text{for } \varphi \in C_0^\infty(Q(T))^2. \end{aligned}$$

Thus, (3.3.3) implies that

$$\int_{Q(T)} u \cdot \varphi dx dt = 0 \quad \text{for } \varphi \in C_0^\infty(Q(T))^2,$$

and $u = 0$ on $Q(T)$. Hence, we have proved the uniqueness of the solution for P_0 . \square

3.4 Existence

In this section we prove existence of a solution to P_0 . Since V_1 is a separable Hilbert space, we can choose a complete orthonormal system $\{\psi_n\}_{n=1}^\infty$ of V_1 normalized in H . Also, we shall use the closed linear space V_{1n} generated by $\psi_1, \psi_2, \dots, \psi_n$ for $n \in \mathbb{Z}_{>0}$. Moreover, since $u_0 \in V_1$, $v_0 \in H$ and V_1 is dense in H , there exist $\{u_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V_1$, $\{v_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V_1$ and $\{m_n\} \subset \{m\}$ such that

$$\begin{aligned} u_{0n}, v_{0n} &\in V_{1m_n} \text{ for } n \in \mathbb{Z}_{>0}, \\ u_{0n} &\rightarrow u_0 \text{ in } V_1 \text{ and } v_{0n} \rightarrow v_0 \text{ in } H \text{ and } m_n \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

We prove the existence by the Galerkin method, namely, first for $n \in \mathbb{Z}_{>0}$ we find $u_n(t) = \sum_{k=1}^{m_n} a_k^{(n)}(t) \psi_k$ satisfying

$$\begin{aligned} \rho \int_0^1 u_{ntt}(t) \cdot \psi_j dx + \gamma \int_0^1 u_{ntt}(t) \cdot \psi_{jxx} dx \\ + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot \psi_{jx} dx = 0 \text{ for } t \in [0, T] \text{ and } j = 1, 2, \dots, m_n, \end{aligned} \quad (3.4.1)$$

$$u_n(0) = u_{0n}, u_{nt}(0) = v_{0n} \text{ and } \varepsilon_n = |u_{nx}| - 1 \text{ on } Q(T). \quad (3.4.2)$$

We denote by $P_{0n}(u_{0n}, v_{0n})$ the problem (3.4.1) and (3.4.2) for each $n \in \mathbb{Z}_{>0}$. For proving the existence of a solution u_n of $P_{0n}(u_{0n}, v_{0n})$ for $n \in \mathbb{Z}_{>0}$, we solve the following initial value problem $I_{0n}(a_0^{(n)}, b_0^{(n)})$ for the ordinary differential equations:

Find $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \dots, a_{m_n}^{(n)}) \in C^2([0, T])^{m_n}$ such that

$$\begin{aligned} \rho \frac{d^2 a^{(n)}}{dt^2} &= -F(a^{(n)}) - G(a^{(n)}) \text{ on } [0, T], \\ a^{(n)}(0) &= a_0^{(n)}, \frac{da^{(n)}}{dt} = b_0^{(n)}, \end{aligned}$$

where $a_0^{(n)} = (a_{01}^{(n)}, a_{02}^{(n)}, \dots, a_{0m_n}^{(n)}) \in \mathbb{R}^{m_n}$, $b_0^{(n)} = (b_{01}^{(n)}, b_{02}^{(n)}, \dots, b_{0m_n}^{(n)}) \in \mathbb{R}^{m_n}$, $F = (F_1, F_2, \dots, F_{m_n})$, $G = (G_1, G_2, \dots, G_{m_n})$,

$$\begin{aligned} F_j(a^{(n)}) &= \gamma \sum_{k=1}^{m_n} a_k^{(n)}(t) \int_0^1 \psi_{kxx} \cdot \psi_{jxx} dx, \\ G_j(a^{(n)}) &= \int_0^1 f \left(\left| \sum_{k=1}^{m_n} a_k^{(n)}(t) \psi_{kx}(x) \right| - 1 \right) \left(\sum_{k=1}^{m_n} a_k^{(n)}(t) \psi_{kx}(x) \cdot \psi_{jx}(x) \right) dx \\ &\quad \text{for } j = 1, 2, \dots, m_n. \end{aligned}$$

The existence and uniqueness of the solution for I_n is guaranteed by the Banach fixed point theorem. Thus, we have:

Lemma 3.8. *Let $n \in \mathbb{Z}_{>0}$. If $a_0^{(n)} = (a_{01}^{(n)}, a_{02}^{(n)}, \dots, a_{0m_n}^{(n)}) \in \mathbb{R}^{m_n}$, $b_0^{(n)} = (b_{01}^{(n)}, b_{02}^{(n)}, \dots, b_{0m_n}^{(n)}) \in \mathbb{R}^{m_n}$ satisfying $u_{0n} = \sum_{k=1}^{m_n} a_{0k}^{(n)} \psi_k$, $v_{0n} = \sum_{k=1}^{m_n} b_{0k}^{(n)} \psi_k$, then there exists one and only one $u_n \in$*

$C^2([0, T]; V_{1m_n})$ satisfying (3.4.1) and (3.4.2). Also, it holds that

$$\rho \int_0^1 u_{ntt}(t) \cdot \eta dx + \gamma \int_0^1 u_{nxx}(t) \cdot \eta_{xx} dx + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot \eta_x dx = 0 \text{ for } \eta \in V_{1m_n} \text{ for } t \in [0, T]. \quad (3.4.3)$$

Now, we give a lemma concerned with the uniform estimate of u_n .

Lemma 3.9. *If u_n is a solution of P_{0n} on $[0, T]$ for $n \in \mathbb{Z}_{>0}$, the following energy G_n is conserved:*

$$G_n = \frac{\rho}{2} \int_0^1 |u_{nt}|^2 dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 dx + \frac{1}{2} \int_0^1 \widehat{g}(|u_{nx}|^2) dx, \quad \frac{d}{dt} G_n = 0 \text{ on } [0, T],$$

where \widehat{g} is a primitive of f , and satisfies $\widehat{g}(1) = 0$. Moreover, it holds that

$$\frac{\rho}{2} \int_0^1 |u_{nt}|^2 dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 dx \leq G_n(0) \text{ on } [0, T].$$

Proof. Let u_n be a solution of P_{0n} on $[0, T]$, namely, it is represented by $u_n = \sum_{k=1}^{m_n} a_k^{(n)} \psi_k$ on $Q(T)$ for $n \in \mathbb{Z}_{>0}$. By substituting $\eta = u_{nt}$ into (3.4.3), we have

$$\begin{aligned} \rho \int_0^1 u_{ntt}(t) \cdot u_{nt}(t) dx + \gamma \int_0^1 u_{nxx}(t) \cdot u_{ntxx}(t) dx \\ + \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot u_{ntx}(t) dx = 0 \text{ for } t \in [0, T]. \end{aligned}$$

Here, we put $z = |u_{nx}|^2$ and $g(z) = f(\sqrt{z} - 1)$ for $z \in \mathbb{R}$, and then we have

$$\begin{aligned} \int_0^1 f(\varepsilon_n(t)) u_{nx}(t) \cdot u_{ntx}(t) dx &= \frac{1}{2} \int_0^1 \frac{\partial}{\partial t} \widehat{g}(z) dx \\ &= \frac{1}{2} \frac{d}{dt} \int_0^1 \widehat{g}(|u_{nx}|^2) dx \text{ for } t \in [0, T], \end{aligned}$$

where $\widehat{g}(r) = \int_1^r g(\xi) d\xi$ for $r \in \mathbb{R}$. Hence, we obtain

$$\frac{d}{dt} \left(\frac{\rho}{2} \int_0^1 |u_{nt}|^2 dx + \frac{\gamma}{2} \int_0^1 |u_{nxx}|^2 dx + \frac{1}{2} \int_0^1 \widehat{g}(|u_{nx}|^2) dx \right) = 0 \text{ on } [0, T].$$

Clearly, \widehat{g} is a primitive of g and satisfies $\widehat{g}(1) = 0$. Since f is monotone increasing, we see that $\widehat{g}(r) \geq 0$ for any $r \in \mathbb{R}$. Thus, Lemma 3.9 has been proved. \square

Lemma 3.10. *It holds that $\{u_n\}_{n \in \mathbb{Z}_{>0}}$ is bounded in $L^\infty(0, T; V_1)$ and $W^{1\infty}(0, T; H)$.*

Proof. First, by the boundedness of $\{u_{0n}\}_{n \in \mathbb{Z}_{>0}}$ in V_1 , $\{u_{0nx}\}_{n \in \mathbb{Z}_{>0}}$ is bounded in $L^\infty(0, 1)^2$. This shows that $\{G_n(0)\}_{n \in \mathbb{Z}_{>0}}$ is bounded and $\{u_{nt}\}_{n \in \mathbb{Z}_{>0}}$ and $\{u_{nxx}\}_{n \in \mathbb{Z}_{>0}}$ are bounded in $L^\infty(0, T; H)$. Hence, it is clear that the assertion of this lemma is true. \square

Next, we show existence of a convergence subsequence. Here, we put $X = \{z \in H^1(0, 1)^2 | z(0) = z(1)\}$, again.

Lemma 3.11. *There exist a subsequence $\{n_k\} \subset \{n\}$ and a function u on $Q(T)$ such that $u \in L^\infty(0, T; V_1) \cap W^{1,\infty}(0, T; H)$,*

$u_{n_k} \rightarrow u$ weakly in $L^\infty(0, T; V_1)$, in $L^2(0, T; X)$ and weakly* in $W^{1,\infty}(0, T; H)$ as $k \rightarrow \infty$.*

Proof. By Lemma 3.10 and Aubin's compact theorem (cf. [20]), it is easy to show existence of the subsequence with the required condition. \square

The following lemma is concerned with approximation of the test function η .

Lemma 3.12. *For $\eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V_1)$ with $\eta(T) = 0$, there exists $\{\eta_n\} \subset W^{1,2}(0, T; V_1)$ such that*

$$\begin{aligned} \eta_n &\in L^2(0, T; V_{1n}), \eta_n(T) = 0, \eta_n(0) \rightarrow \eta(0) \text{ in } H \text{ as } n \rightarrow \infty, \\ \eta_n &\rightarrow \eta \text{ in } L^2(0, T; V_1) \text{ and } \eta_{nt} \rightarrow \eta_t \text{ in } L^2(0, T; H) \text{ as } n \rightarrow \infty. \end{aligned}$$

Proof of the existence. Put $u_k = u_{n_k}$ and $\eta_k = \eta_{n_k}$ for $k \in \mathbb{Z}_{>0}$. Since u_k is the solution of P_{0n_k} , by Lemma 3.8 we obtain

$$\rho \int_{Q(T)} u_{ktt} \cdot \eta_k dxdt + \gamma \int_{Q(T)} u_{kxx} \cdot \eta_{kxx} dxdt + \int_{Q(T)} f(\varepsilon_{n_k}) u_{kx} \cdot \eta_{kx} dxdt = 0,$$

and

$$\begin{aligned} &-\rho \int_{Q(T)} u_{kt} \cdot \eta_k dxdt + \gamma \int_{Q(T)} u_{kxx} \cdot \eta_{kxx} dxdt + \int_{Q(T)} f(\varepsilon_{n_k}) u_{kx} \cdot \eta_{kx} dxdt \\ &= - \int_0^1 v_{0n_k} \eta_k(0) dx \text{ for } k \in \mathbb{Z}_{>0}. \end{aligned}$$

By letting $k \rightarrow \infty$ in this equation, Lemmas 3.11 and 3.12 guarantee that u satisfies the condition in Definition 3.1. Hence, the existence of the solution to $P_0(u_0, v_0)$ has been proved. \square

3.5 Numerical results

In this section, we show numerical results for P_0 on the domain $(0, T) \times (0, l_*)$. The purpose of our numerical simulation is to clarify the role of the term γu_{xxxx} in (1.3.8). For this purpose, we provide graphs of approximate solutions in varying the parameter γ . We choose a grid $x_j = (j-1)\Delta x$, $j = 1, 2, \dots, N$, with $\Delta x = l_*/N$ and we denote the approximation of $u(t, x_j)$ by $u_j(t)$. The fourth derivative in (1.3.8) is approximated using

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4}(t, x_j) &\approx \frac{u_{xx}(t, x_{j+1}) - 2u_{xx}(t, x_j) + u_{xx}(t, x_{j-1}))}{(\Delta x)^2} \\ &\approx \frac{\frac{u_{j+2} - 2u_{j+1} + u_j}{(\Delta x)^2} - 2\frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} + \frac{u_j - 2u_{j-1} + u_{j-2}}{(\Delta x)^2}}{(\Delta x)^2} \end{aligned}$$

$$= \frac{u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{(\Delta x)^4}. \quad (3.5.1)$$

The other spatial derivative in (1.3.8) is approximated using central differences, so

$$\frac{\partial}{\partial x} \left(f(\varepsilon) \frac{\partial u}{\partial x} \right) (t, x_j) \approx \frac{(f(\varepsilon) \frac{\partial u}{\partial x})(t, x_{j+1}) - (f(\varepsilon) \frac{\partial u}{\partial x})(t, x_{j-1})}{2\Delta x}, \quad (3.5.2)$$

$$\frac{\partial u}{\partial x}(t, x_j) \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}. \quad (3.5.3)$$

The approximation (3.5.3) is also used for the computation of ε . We have now discretized the partial differential equation (1.3.8) in space.

We introduce

$$v(t, x) = \frac{\partial u}{\partial t}(t, x),$$

and denote the approximation of $v(t, x_j)$ by $v_j(t)$. Substituting the approximations (3.5.1)–(3.5.3) into (1.3.8), we find the first order ordinary differential equation system ($j = 1, 2, \dots, N$)

$$\begin{aligned} u_j' &= v_j, \\ v_j' &= \frac{1}{\rho} \left(\frac{-\gamma}{(\Delta x)^4} (u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}) \right. \\ &\quad \left. + \frac{f(\varepsilon_{j+1})(u_{j+2} - u_j) - f(\varepsilon_{j-1})(u_j - u_{j-2})}{4(\Delta x)^2} \right). \end{aligned}$$

The periodic boundary computations (1.3.9) are implemented as $u_{-1} = u_{N-1}$, $u_0 = u_N$, $u_{N+1} = u_1$, $u_{N+2} = u_2$.

The ODE system can be solved using standard methods for time integration. In the numerical experiments below we have used Matlab's `ode45` routine which uses the explicit Runge-Kutta (4,5) pair.

Here, we list the values of parameters: $N = 12$, $l_* = 2\pi$, $\rho = 100$,

$$f(\varepsilon) = \frac{\kappa}{2} \left(\varepsilon + \frac{1}{2} - \frac{1}{2(1 + \varepsilon)^2} \right),$$

$\kappa = 0.1$, $u(0, x) = 0.6(\cos x, \sin x)$, $v(0, x) = (0, 0)$ for $x \in (0, 2\pi)$.

In the following figures, we give graphs of R_i where

$$R_i(t) = \left| u \left(t, \frac{2\pi}{N} i \right) \right| \quad \text{for } i = 0, 1, \dots, N-1.$$

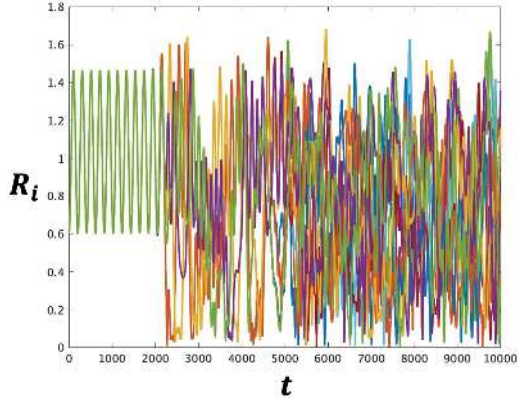


Figure 3.1: $\gamma = 0.0001$

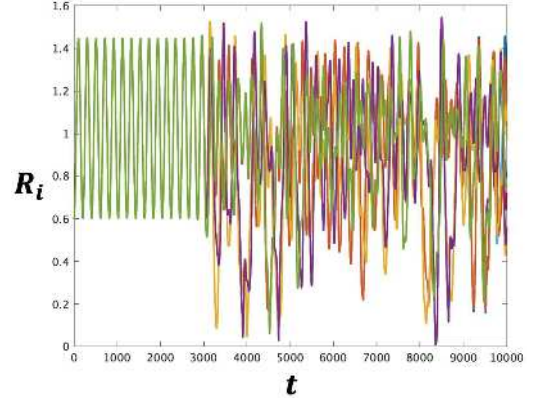


Figure 3.2: $\gamma = 0.001$

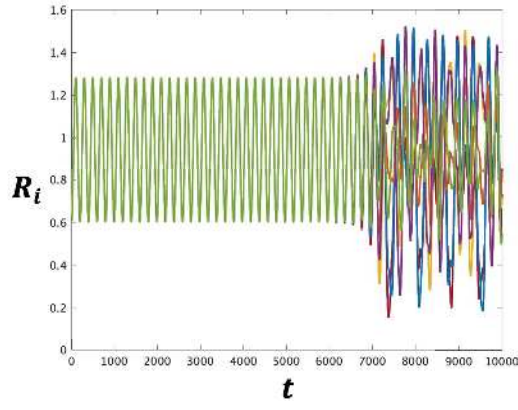


Figure 3.3: $\gamma = 0.01$

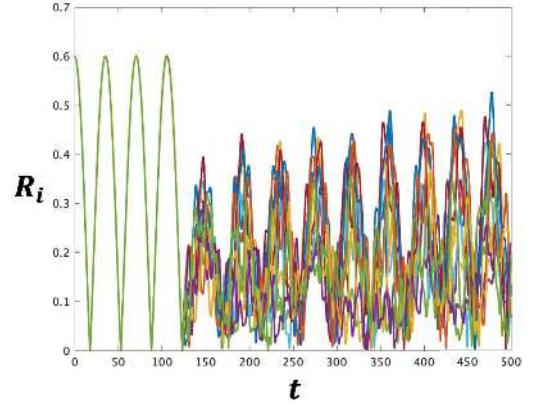


Figure 3.4: $\gamma = 1.0$

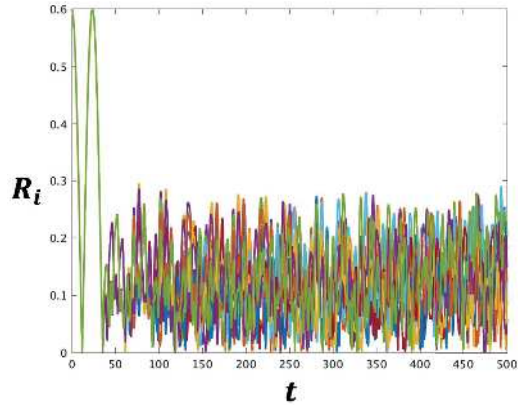


Figure 3.5: $\gamma = 2.0$

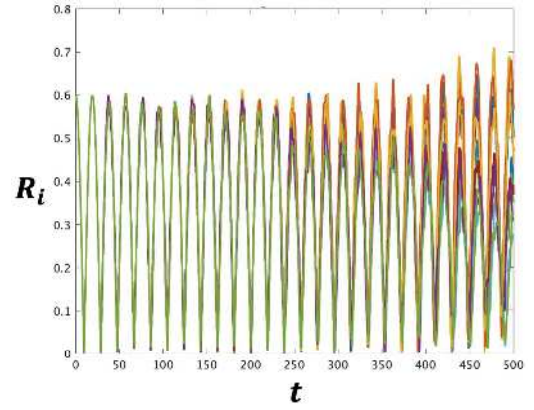


Figure 3.6: $\gamma = 3.0$

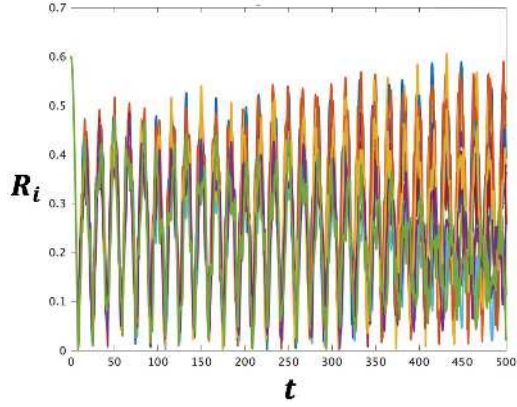


Figure 3.7: $\gamma = 4.0$

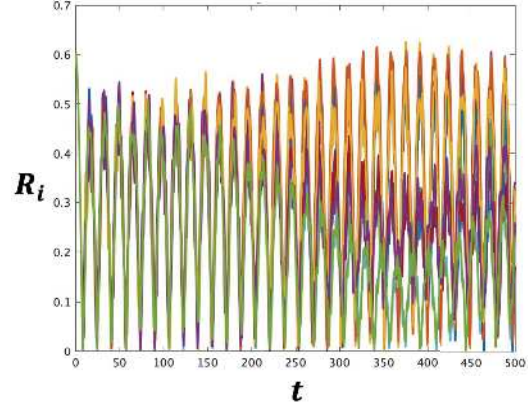


Figure 3.8: $\gamma = 4.1$

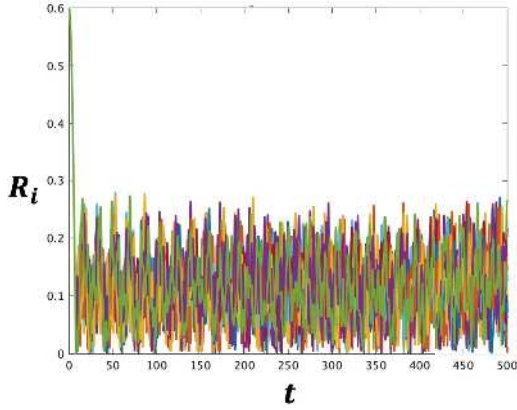


Figure 3.9: $\gamma = 4.3$

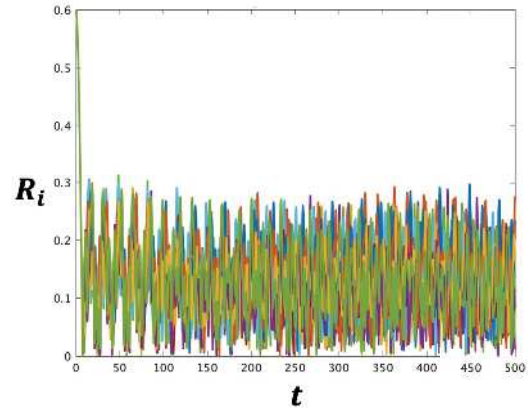


Figure 3.10: $\gamma = 4.33$

In the simulations above, we impose symmetry for the initial condition. In this case we consider that the solution must be periodic in time. However, we do not prove it, yet. For these results obtained by numerical calculations, we give some remarks from two points of view.

The 1st remark is concerned with the relationship between the value of γ and accuracy of our numerical method. From Figures 3.1–3.10 we observe that the solution for $\gamma = 0.01$ exhibits periodicity in longer time than other cases. In particular, for $\gamma > 4$ little periodicity is observed. From these observations we conjecture that an optimal value of γ exists for numerical stability.

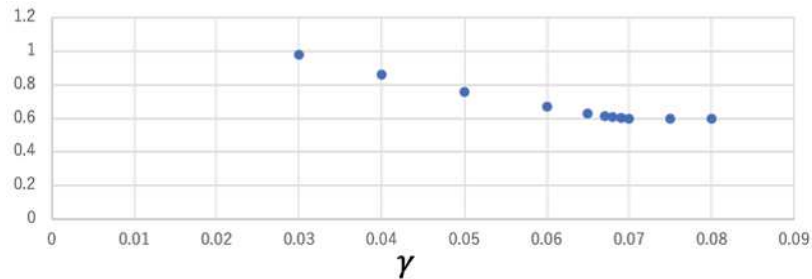


Figure 3.11: Maximum radius vs. γ

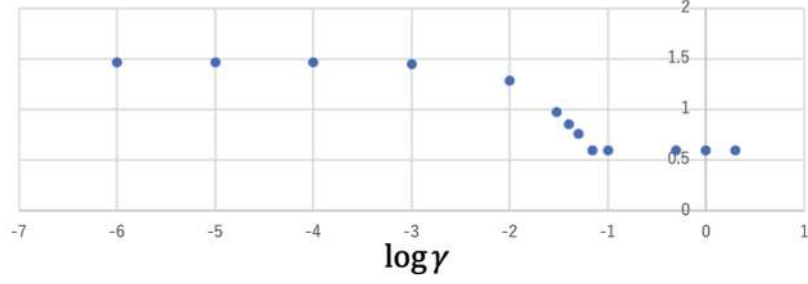


Figure 3.12: Maximum radius vs. $\log \gamma$

The second remark is on the effect of γ on the behavior of the solution. As easily seen, the following energy is conserved:

$$\frac{\rho}{2} \int_0^1 |u_t(t, x)|^2 dx + \frac{\gamma}{2} \int_0^1 |u_{xx}(t, x)|^2 dx + \frac{1}{2} \int_0^1 \hat{f}(|u_x(t, x)|^2) dx \quad \text{for all } t \in [0, T],$$

where

$$\hat{f}(r) = \frac{\kappa}{4} \left(\frac{2}{3} r^{\frac{3}{2}} - \frac{1}{2} r - \frac{1}{2} \log r \right) \quad \text{for } r > 0.$$

Accordingly, we suppose that adding the fourth derivative term make the curve closer to straight. However, from the results plotting γ and $\max_{i, t \in [0, T]} R_i(t)$, and $\log \gamma$ and $\max_{i, t \in [0, T]} R_i(t)$ in Figures 3.11 and 3.12, respectively, our conjecture does not seem to be true. Actually, for the larger γ , the radius becomes smaller, namely, the curve goes farther away from the straight line.

Chapter 4

The partial differential equation model with stress function having singularity

4.1 Main results

We recall that

$$\begin{aligned} H &:= L^2(0, 1)^2, X := \{z \in H^1(0, 1)^2 \mid z(0) = z(1)\}, \\ V_1 &:= \{z \in H^2(0, 1)^2 \mid z(0) = z(1), z_x(0) = z_x(1)\}, \\ V_2 &:= \{z \in H^4(0, 1)^2 \mid z(0) = z(1), z_x(0) = z_x(1), z_{xx}(0) = z_{xx}(1), z_{xxx}(0) = z_{xxx}(1)\} \end{aligned}$$

with the standard norm denoted by $|\cdot|_H := |\cdot|_{L^2(0,1)^2}$, $|\cdot|_X := |\cdot|_{H^1(0,1)^2}$, $|\cdot|_{V_1} := |\cdot|_{H^2(0,1)^2}$ and $|\cdot|_{V_2} := |\cdot|_{H^4(0,1)^2}$, and throughout this chapter, we assume $\mu > 0$.

Here, we define strong and weak solutions of P_μ given by (1.3.11)–(1.3.13) for $\mu > 0$.

Definition 4.1. A function u from $Q(T)$ to \mathbb{R}^2 is called a strong solution of P_μ on $Q(T)$ if $u \in W^{1,\infty}(0, T; V_1) \cap L^\infty(0, T; H^4(0, 1)^2) \cap W^{2,2}(0, T; H) \cap W^{1,2}(0, T; H^3(0, 1)^2)$, $|u_x| > 0$ on $\overline{Q(T)}$, and satisfies (1.3.11) - (1.3.13) in the usual sense.

Definition 4.2. A function u from $Q(T)$ to \mathbb{R}^2 is called a weak solution of P_μ on $Q(T)$ if u satisfies the following conditions: $u \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V_1) \cap W^{1,2}(0, T; H^1(0, 1)^2)$, $u(0) = u_0$, $|u_x| > 0$ on $Q(T)$ and

$$\begin{aligned} & -\rho \int_{Q(T)} u_t \cdot \eta_t dxdt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dxdt + \int_{Q(T)} f(\varepsilon) u_x \cdot \eta_x dxdt + \mu \int_{Q(T)} u_{tx} \cdot \eta_x dxdt \\ & = \int_0^1 v_0 \cdot \eta(0) dx \text{ for } \eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V_1) \text{ with } \eta(T) = 0 \text{ a.e. on } (0, 1). \end{aligned}$$

We note that $u \cdot v = u_1 v_1 + u_2 v_2$ for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$. The main results of this chapter are as follows:

Theorem 4.1. Let $\rho, \gamma, \mu, T > 0$. If f is given by (1.3.14), $u_0 \in V_1$, $|u_{0x}| > 0$ on $[0, 1]$ and $v_0 \in H$, then P_μ has one and only one weak solution on $Q(T)$.

Theorem 4.2. *Let $\rho, \gamma, \mu, T > 0$. If f is given by (1.3.14), $u_0 \in V_2$, $|u_{0x}| > 0$ on $[0, 1]$, and $v_0 \in V_1$, then P_μ has a strong solution on $Q(T)$.*

The proof of the uniqueness of weak solutions is given in the next section. In Section 4.3 we prove the existence of strong solutions, and in Section 4.4 we prove the existence of weak solutions. Since a strong solution is clearly a weak solution, the uniqueness of the strong solution holds.

4.2 Uniqueness

In this section we give a proof of the uniqueness for a solution to P_μ on $Q(T)$ and suppose all assumptions of Theorem 4.1.

Before proving Theorem 4.1, we consider the following linear problem $\bar{P}_\mu := \bar{P}_\mu(u_0, v_0, F)$: Find $u : Q(T) \rightarrow \mathbb{R}^2$ satisfying

$$\rho u_{tt} + \gamma u_{xxxx} - \mu u_{txx} = F_x \quad \text{on } Q(T), \quad (4.2.1)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \quad \text{on } (0, T), \text{ for all } i = 0, 1, 2, 3, \quad (4.2.2)$$

$$u(0) = u_0, \quad \frac{\partial}{\partial t} u(0) = v_0 \quad \text{on } (0, 1), \quad (4.2.3)$$

where F is a given function on $Q(T)$. Here, we define a weak solution of \bar{P}_μ in the following way.

Definition 4.3. A function u from $Q(T)$ to \mathbb{R}^2 is called a weak solution of \bar{P}_μ on $Q(T)$ if u satisfies the following conditions: $u \in W^{1,2}(0, T; H^1(0, 1)^2) \cap L^2(0, T; V_1)$, $u(0) = u_0$ and

$$\begin{aligned} & -\rho \int_{Q(T)} u_t \cdot \eta_t dx dt + \gamma \int_{Q(T)} u_{xx} \cdot \eta_{xx} dx dt + \int_{Q(T)} F \cdot \eta_x dx dt + \mu \int_{Q(T)} u_{tx} \cdot \eta_x dx dt \\ & = \rho \int_0^1 v_0 \cdot \eta(0) dx \quad \text{for } \eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V_1) \text{ with } \eta(T) = 0 \text{ a.e. on } (0, 1). \end{aligned}$$

Lemma 4.1.

(1) *If $u_0 \in V_1$, $v_0 \in H$, and $F \in L^2(0, T; H)$, then $\bar{P}_\mu(u_0, v_0, F)$ has a unique weak solution on $Q(T)$.*

(2) *If $u_0 \in V_2$, $v_0 \in V_1$, and $F \in L^2(0, T; V_1)$, then $\bar{P}_\mu(u_0, v_0, F)$ has a strong solution u on $Q(T)$, namely, $u \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; H^2(0, 1)^2) \cap W^{1,2}(0, T; H^3(0, 1)^2) \cap L^\infty(0, T; H^4(0, 1)^2)$. Moreover, it holds that*

$$\begin{aligned} & \frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \\ & \leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx}|_H^2 + \frac{1}{\mu} \int_0^t |F_{xx}|_H^2 d\tau \quad \text{for any } t \in [0, T]. \end{aligned} \quad (4.2.4)$$

and

$$\begin{aligned} & \frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}|_H^2 d\tau \\ & \leq \frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{1}{\mu} \int_0^t |F|_H^2 d\tau \quad \text{for any } t \in [0, T]. \end{aligned} \quad (4.2.5)$$

(3) For $u_{01}, u_{02} \in V_2$, $v_{01}, v_{02} \in V_1$, and $F_1, F_2 \in L^2(0, T; V_1)$, let u_1 and u_2 be strong solutions of $\bar{P}_\mu(u_{01}, v_{01}, F_1)$ and $\bar{P}_\mu(u_{02}, v_{02}, F_2)$, respectively. Then it holds that

$$\begin{aligned} & \frac{\rho}{2}|u_{1t}(t) - u_{2t}(t)|_H^2 + \frac{\gamma}{2}|u_{1xx}(t) - u_{2xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{1\tau x} - u_{2\tau x}|_H^2 d\tau \\ & \leq \frac{\rho}{2}|v_{01} - v_{02}|_H^2 + \frac{\gamma}{2}|u_{01xx} - u_{02xx}|_H^2 + \frac{1}{2\mu} \int_0^t |F_1 - F_2|_H^2 d\tau \text{ for any } t \in [0, T]. \end{aligned} \quad (4.2.6)$$

In the similarly way to the proof given in Chapter 3 in this thesis and [6, Section 3], we can show the uniqueness of weak solutions for \bar{P}_μ . Also, we can prove the existence of strong solutions for \bar{P}_μ and (4.2.6), similarly. So we omit its proof.

Lemma 4.2. Let $u_{01}, u_{02} \in V_1$, $v_{01}, v_{02} \in H$, $F_1, F_2 \in L^2(0, T; H)$. If u_1 and u_2 are weak solutions of $\bar{P}_\mu(u_{01}, v_{01}, F_1)$ and $\bar{P}_\mu(u_{02}, v_{02}, F_2)$ on $Q(T)$, respectively, then (4.2.6) holds for them.

Proof. Since $u_{0i} \in V_1$, $v_{0i} \in H$, and $F_i \in L^2(0, T; H)$ for $i = 1, 2$, we can take the following sequences:

$$\begin{aligned} & \{u_{0in}\}_{n \in \mathbb{Z}_{>0}} \subset V_2 \text{ such that } u_{0in} \rightarrow u_{0i} \text{ in } V_1 \text{ as } n \rightarrow \infty, \\ & \{v_{0in}\}_{n \in \mathbb{Z}_{>0}} \subset V_1 \text{ such that } v_{0in} \rightarrow v_{0i} \text{ in } H \text{ as } n \rightarrow \infty, \\ & \{F_{in}\}_{n \in \mathbb{Z}_{>0}} \subset L^2(0, T; V_1) \text{ such that } F_{in} \rightarrow F_i \text{ in } L^2(0, T; H) \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.2.7)$$

By using Lemma 4.1, for $i = 1, 2$ and all $n \in \mathbb{Z}_{>0}$, there exists a strong solution u_{in} of $\bar{P}_\mu(u_{0in}, v_{0in}, F_{in})$ on $Q(T)$.

Let $n, m \in \mathbb{Z}_{>0}$. By using (4.2.6) in Lemma 4.1, we have

$$\begin{aligned} & \frac{\rho}{2}|u_{int}(t) - u_{imt}(t)|_H^2 + \frac{\gamma}{2}|u_{inxx}(t) - u_{imxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{in\tau x} - u_{im\tau x}|_H^2 d\tau \\ & \leq \frac{\rho}{2}|v_{0in} - v_{0im}|_H^2 + \frac{\gamma}{2}|u_{0inxx} - u_{0imxx}|_H^2 + \frac{1}{2\mu} \int_0^t |F_{in} - F_{im}|_H^2 d\tau \quad \text{for any } t \in [0, T]. \end{aligned}$$

From (4.2.7) it is clear that $\{u_{in}\}$ is a Cauchy sequence in $C^1([0, T]; H)$, $C([0, T]; H^2(0, 1)^2)$ and $W^{1,2}(0, T; H^1(0, 1)^2)$ for each i . Hence, for $i = 1, 2$, there exists $u_i \in C^1([0, T]; H) \cap C([0, T]; H^2(0, 1)^2) \cap W^{1,2}(0, T; H^1(0, 1)^2)$ such that

$$u_{in} \rightarrow u_i \text{ in } C^1([0, T]; H), C([0, T]; H^2(0, 1)^2) \text{ and } W^{1,2}(0, T; H^1(0, 1)^2) \text{ as } n \rightarrow \infty.$$

Obviously, these convergences imply that u_i is a weak solution of $\bar{P}_\mu(u_{0i}, v_{0i}, F_i)$ on $Q(T)$ for $i = 1, 2$. On account of the above convergences and (4.2.7), we get (4.2.6). Thus, Lemma 4.2 has been proved. \square

Proof of the uniqueness of weak solutions to P_μ . Let u_1 and u_2 be weak solutions for P_μ on $Q(T)$. Easily, we see that for $i = 1, 2$, u_i is a weak solution of $\bar{P}_\mu(u_0, v_0, F_i)$ on $Q(T)$, where $F_i = f(\varepsilon_i)u_{ix}$, $\varepsilon_i = |u_{ix}| - 1$. By using Lemma 4.2, we have

$$\begin{aligned}
& \frac{\rho}{2}|u_{1t}(t) - u_{2t}(t)|_H^2 + \frac{\gamma}{2}|u_{1xx}(t) - u_{2xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{1\tau x} - u_{2\tau x}|_H^2 d\tau \\
& \leq \frac{1}{2\mu} \int_0^t |f(\varepsilon_1)u_{1x} - f(\varepsilon_2)u_{2x}|_H^2 d\tau \\
& \leq \frac{1}{\mu} \left(\int_0^t |(f(\varepsilon_1) - f(\varepsilon_2))u_{1x}|_H^2 d\tau + \int_0^t |f(\varepsilon_2)(u_{1x} - u_{2x})|_H^2 d\tau \right) \\
& \leq \frac{1}{\mu} \left(|u_{1x}|_{L^\infty(Q(T))}^2 \int_0^t |f(\varepsilon_1) - f(\varepsilon_2)|_H^2 d\tau + \int_0^t |f(\varepsilon_2)(u_{1x} - u_{2x})|_H^2 d\tau \right) \\
& \quad \text{for any } n, m \in \mathbb{Z}_{>0} \text{ and } t \in [0, T]. \tag{4.2.8}
\end{aligned}$$

Since $u_{ix} \in C(\overline{Q(T)})$ and $|u_{ix}| > 0$ on $\overline{Q(T)}$ for $i = 1, 2$, there exists $C > 0$ such that $|u_{1x}| \leq C$, $|f(\varepsilon_2)| \leq C$, and $|f(\varepsilon_1) - f(\varepsilon_2)| \leq C|\varepsilon_1 - \varepsilon_2|$ on $\overline{Q(T)}$. Accordingly, from (4.2.8) we see that

$$\begin{aligned}
& \frac{\rho}{2}|u_{1t}(t) - u_{2t}(t)|_H^2 + \frac{\gamma}{2}|u_{1xx}(t) - u_{2xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{1\tau x} - u_{2\tau x}|_H^2 d\tau \\
& \leq \frac{1}{\mu} \left(C^4 \int_0^t |\varepsilon_1 - \varepsilon_2|_H^2 d\tau + C^2 \int_0^t |u_{1x} - u_{2x}|_H^2 d\tau \right) \\
& \leq \frac{C^2}{\mu} (C^2 + 1) \int_0^t |u_{1x} - u_{2x}|_H^2 d\tau \\
& \leq \frac{C^2}{2\mu} (C^2 + 1) \left(\int_0^t |u_1 - u_2|_H^2 d\tau + \int_0^t |u_{1xx} - u_{2xx}|_H^2 d\tau \right) \\
& \leq \frac{C^2}{2\mu} (C^2 + 1) \left(T^2 \int_0^t |u_{1t} - u_{2t}|_H^2 d\tau + \int_0^t |u_{1xx} - u_{2xx}|_H^2 d\tau \right) \text{ for any } t \in [0, T]. \tag{4.2.9}
\end{aligned}$$

By applying the Gronwall inequality to (4.2.9), we obtain

$$\frac{\rho}{2}|u_{1t}(t) - u_{2t}(t)|_H^2 + \frac{\gamma}{2}|u_{1xx}(t) - u_{2xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{1\tau x} - u_{2\tau x}|_H^2 d\tau = 0 \text{ for } t \in [0, T].$$

This together with the initial condition implies $u_1 = u_2$ on $Q(T)$. Thus, the uniqueness of the weak solution to P_μ is proved. \square

4.3 Existence of strong solutions

In this section we give a proof for existence of strong solutions to P_μ . The following Lemmas 4.3 - 4.5 are keys in our proof of the existence of strong solutions.

Lemma 4.3. *Let $z \in H^2(0, 1)^2$ and $K_1, K_2 > 0$. If*

$$\int_0^1 \frac{1}{|z_x|^2} dx \leq K_1, |z_{xx}|_H \leq K_2,$$

then we obtain

$$|z_x| \geq \frac{K_2}{\sqrt{2}} e^{-K_1 K_2^2} \quad \text{on } [0, 1].$$

By using a same way as in the proof of Lemma 3.2 of Section 3 in [8], we can show this lemma. So, we omit its proof.

In order to apply the fixed point theorem, we introduce a set as follows. For $T > 0$, $\delta > 0$, $M > 0$, $M' > 0$ and $\overline{M} > 0$, we denote $K(T, \delta, M, M', \overline{M})$ the set of v satisfying the following conditions (C1) - (C5):

$$(C1) \quad v \in L^2(0, T; H^3(0, 1)^2) \cap L^\infty(0, T; L^\infty(0, 1)^2).$$

$$(C2) \quad \delta \leq |v_x| \leq M \text{ a.e. on } Q(T).$$

$$(C3) \quad |v_{xx}(t)|_H \leq M' \text{ for a.e. } t \in (0, T).$$

$$(C4) \quad \frac{\partial^i}{\partial x^i} v(t, 0) = \frac{\partial^i}{\partial x^i} v(t, 1) \text{ for a.e. } t \in (0, T) \text{ and for } i = 1, 2, 3.$$

$$(C5) \quad \int_0^T |v_{xxx}(t)|_H^2 dt \leq \overline{M}.$$

We define the norm of $W(T) := L^2(0, T; H^3(0, 1)^2) \cap L^\infty(0, T; L^\infty(0, 1)^2)$ by

$$|z|_{W(T)} := \left(|z|_{L^2(0, T; H^3(0, 1)^2)}^2 + |z_x|_{L^\infty(0, T; L^\infty(0, 1)^2)}^2 \right)^{\frac{1}{2}}.$$

Clearly, $K(T, \delta, M, M', \overline{M})$ is closed in $W(T)$ and for $v \in K(T, \delta, M, M', \overline{M})$, $f(\hat{\varepsilon})v_x \in L^2(0, T; V_1)$. By using Lemma 4.1, we can define $\Lambda : K(T, \delta, M, M', \overline{M}) \rightarrow W(T)$ by $\Lambda v = u$, where u is a strong solution of $\overline{P}_\mu(u_0, v_0, f(\hat{\varepsilon})v_x)$ on $Q(T)$ with $\hat{\varepsilon} = |v_x| - 1$.

Lemma 4.4. *If $u_0 \in V_2$, $|u_{0x}| > 0$ on $[0, 1]$ and $v_0 \in V_1$, then there exist positive constants $T_1, \delta, M, M', \overline{M}$ such that $\Lambda : K(T_1, \delta, M, M', \overline{M}) \rightarrow K(T_1, \delta, M, M', \overline{M})$ is a contraction.*

We give a proof of Lemma 4.4 after the proof of Lemma 4.5.

Lemma 4.5. *For $T_0 \in (0, T]$, $\delta, M, M', \overline{M} > 0$, there exists a positive constant α such that for $v \in K(T_0, \delta, M, M', \overline{M})$ and a strong solution u of $\overline{P}_\mu(u_0, v_0, f(\hat{\varepsilon})v_x)$ on $Q(T_0)$ with $\hat{\varepsilon} = |v_x| - 1$ it holds that*

$$\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx}|_H^2 + \alpha$$

for any $t \in [0, T_0]$. (4.3.1)

Moreover, by putting $\beta = \frac{\kappa^2 M^2}{32\mu} \left(1 + \frac{1}{\delta^4}\right)^2$, it holds that

$$\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}|_H^2 d\tau \leq \frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \beta T_0 \quad \text{for any } t \in [0, T_0]. \quad (4.3.2)$$

Proof. Let $T_0 \in (0, T]$ and $v \in K(T_0, \delta, M, M', \overline{M})$. Since $f(\hat{\varepsilon})v_x \in L^2(0, T_0; V_1)$, the inequality (4.2.4) holds with $F = f(\hat{\varepsilon})v_x$, namely, we see that

$$\begin{aligned} & \frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \\ & \leq \frac{\rho}{2}|v_{0xx}|_H^2 + \frac{\gamma}{2}|u_{0xxxx}|_H^2 + \frac{1}{\mu} \int_0^t |(f(\hat{\varepsilon})v_x)_{xx}|_H^2 d\tau \quad \text{for } t \in [0, T_0]. \end{aligned} \quad (4.3.3)$$

The third term of the right hand side in (4.3.3) is estimated by

$$\begin{aligned} \frac{1}{\mu} \int_0^t |(f(\hat{\varepsilon})v_x)_{xx}|_H^2 d\tau & \leq \frac{1}{\mu} \left(3 \int_0^t |f(\hat{\varepsilon})v_{xxx}|_H^2 d\tau + 12 \int_0^t |f(\hat{\varepsilon})_x v_{xx}|_H^2 d\tau \right. \\ & \quad \left. + 3 \int_0^t |f(\hat{\varepsilon})_{xx} v_x|_H^2 d\tau \right) \\ & =: \frac{1}{\mu} (R_1 + R_2 + R_3) \quad \text{for } t \in [0, T_0]. \end{aligned}$$

By $f(\hat{\varepsilon}) = \frac{\kappa}{4} \left(1 - \frac{1}{(1 + \hat{\varepsilon})^4} \right)$, $\hat{\varepsilon} = |v_x| - 1$ on $Q(T)$, (C2) and (C5), we have

$$\begin{aligned} R_1 & \leq \frac{3\kappa^2}{16} \int_0^t \int_0^1 \left(1 + \frac{1}{|v_x|^4} \right)^2 |v_{xxx}|^2 dx d\tau \\ & \leq \frac{3\kappa}{16} \left(1 + \frac{1}{\delta^4} \right)^2 \int_0^{T_0} |v_{xxx}|_H^2 d\tau \\ & \leq \frac{3\kappa^2 \overline{M}}{16} \left(1 + \frac{1}{\delta^4} \right)^2 \quad \text{for } t \in [0, T_0]. \end{aligned} \quad (4.3.4)$$

Next, by elementary calculations and (C2), it follows that

$$\begin{aligned} R_2 & \leq 12\kappa^2 \int_0^t \int_0^1 \frac{|v_{xx}|^4}{|v_x|^{10}} dx d\tau \\ & \leq \frac{12\kappa^2}{\delta^{10}} \int_0^t \int_0^1 |v_{xx}|^4 dx d\tau \quad \text{for } t \in [0, T_0]. \end{aligned}$$

Here, we note that the following inequality holds:

$$|z|_{L^\infty(0,1)^2} \leq |z|_H + |z_x|_H \quad \text{for } z \in H^1(0, 1)^2. \quad (4.3.5)$$

By using (4.3.5), (C2), (C3) and (C5), we obtain

$$\begin{aligned} R_2 & \leq \frac{12\kappa^2}{\delta^{10}} \int_0^t (|v_{xx}|_H + |v_{xxx}|_H)^2 |v_{xx}|_H^2 d\tau \\ & \leq \frac{24\kappa^2 (M')^2}{\delta^{10}} \left(\int_0^t |v_{xx}|_H^2 d\tau + \int_0^t |v_{xxx}|_H^2 d\tau \right) \\ & \leq \frac{24\kappa^2 (M')^2}{\delta^{10}} \left((M')^2 T_0 + \overline{M} \right) \quad \text{for } t \in [0, T_0]. \end{aligned}$$

In order to estimate R_3 we use that $(f(\hat{\varepsilon}))_{xx} = \kappa \left(\frac{1}{|v_x|^4} + \frac{6|v_x \cdot v_{xx}|^2}{|v_x|^8} + \frac{v_x \cdot v_{xxx}}{|v_x|^6} \right)$ on $Q(T_0)$.

Thanks to (4.3.3), (C2), (C3), (C5) and (4.3.5), we have

$$\begin{aligned}
R_3 &= 3 \int_0^t \int_0^1 |(f(\hat{\varepsilon}))_{xx}|^2 |v_x|^2 dx d\tau \\
&\leq 9\kappa^2 \int_0^t \int_0^1 \left(\frac{1}{|v_x|^6} + \frac{36|v_{xx}|^4}{|v_x|^{10}} + \frac{|v_{xxx}|^2}{|v_x|^8} \right) dx d\tau \\
&\leq 9\kappa^2 \left(\frac{T_0}{\delta^6} + \frac{36}{\delta^{10}} \int_0^t \int_0^1 |v_{xx}(\tau, x)|^4 dx d\tau + \frac{\overline{M}}{\delta^8} \right) \\
&\leq 9\kappa^2 \left(\frac{T_0}{\delta^6} + \frac{36}{\delta^{10}} \int_0^t (|v_{xx}|_H + |v_{xxx}|_H)^2 |v_{xx}|_H^2 d\tau + \frac{\overline{M}}{\delta^8} \right) \\
&\leq 9\kappa^2 \left(\frac{T_0}{\delta^6} + \frac{72(M')^2}{\delta^{10}} \left(\int_0^t |v_{xx}|_H^2 d\tau + \int_0^t |v_{xxx}|_H^2 d\tau \right) + \frac{\overline{M}}{\delta^8} \right) \\
&\leq 9\kappa^2 \left(\frac{T_0}{\delta^6} + \frac{72(M')^2}{\delta^{10}} \left((M')^2 T_0 + \overline{M} \right) + \frac{\overline{M}}{\delta^8} \right) \quad \text{for } t \in [0, T_0].
\end{aligned}$$

Hence, we obtain

$$\frac{1}{\mu} (R_1 + R_2 + R_3) \leq \alpha_1$$

and

$$\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx}|_H^2 + \alpha \text{ for } t \in [0, T_0],$$

where α_1 and α are positive constants.

Finally, in order to show (4.3.2) we consider the third term of the right hand side in (4.2.5). Similarly to (4.3.4), we obtain

$$\begin{aligned}
\frac{1}{2\mu} \int_0^t |f(\hat{\varepsilon})v_x(\tau)|_H^2 d\tau &\leq \frac{\kappa^2}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \int_0^t \int_0^1 |v_x|^2 dx d\tau \\
&\leq \frac{\kappa^2 M^2}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 T_0 \quad \text{for any } t \in [0, T_0],
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}|_H^2 d\tau &\leq \frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 T_0 \\
&\quad \text{for any } t \in [0, T_0].
\end{aligned}$$

Thus, we have proved Lemma 4.5. \square

Proof of Lemma 4.4. First, we prove that Λ is the mapping from $K(T_0, \delta, M, M', \overline{M})$ to $K(T_0, \delta, M, M', \overline{M})$ for some $T_0 \in (0, T]$ and $\delta, M, M', \overline{M} > 0$. Let $T_0 \in (0, T]$, $v \in K(T_0, \delta, M, M', \overline{M})$ and $u = \Lambda v$.

Since there exists a constant $C_1 > 0$ such that

$$|u_x(t, x) - u_x(t', x)| \leq C_1(|u_x(t) - u_x(t')|_H |u_{xx}(t) - u_{xx}(t')|_H)^{\frac{1}{2}} + |u_x(t) - u_x(t')|_H$$

for any $(t, x), (t', x) \in \overline{Q(T)}$,

we have

$$|(\Lambda v)_x(t, x)| \geq |u_{0x}(x)| - \left(C_1 (|u_x(0) - u_x(t)|_H |u_{xx}(0) - u_{xx}(t)|_H)^{\frac{1}{2}} + |u_x(0) - u_x(t)|_H \right)$$

for any $(t, x) \in Q(T_0)$.

By using (4.3.2) in Lemma 4.5, the following uniform estimates hold:

$$\begin{aligned} \int_0^t |u_{\tau x}|_H^2 d\tau &\leq \frac{2}{\mu} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T_0}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right) \quad \text{for any } t \in [0, T_0], \\ |u_{xx}(t)|_H^2 &\leq \frac{2}{\gamma} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T_0}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right) \quad \text{for any } t \in [0, T_0]. \end{aligned}$$

These estimates show that

$$\begin{aligned} |u_x(0) - u_x(t)|_H &\leq \int_0^t |u_{\tau x}|_H d\tau \\ &\leq \sqrt{T_0} \left(\int_0^t |u_{\tau x}|_H^2 d\tau \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{2T_0}{\mu}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \quad \text{for any } t \in [0, T_0] \end{aligned}$$

and

$$\begin{aligned} |u_{xx}(0) - u_{xx}(t)|_H &\leq |u_{0xx}|_H + |u_{xx}(t)|_H \\ &\leq 2\sqrt{\frac{2}{\gamma}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \quad \text{for any } t \in [0, T_0]. \end{aligned}$$

Thus, we have

$$\begin{aligned} |(\Lambda v)_x(t, x)| &\geq |u_{0x}(x)| - \left\{ C_1 (|u_x(0) - u_x(t)| \cdot |u_{xx}(0) - u_{xx}(t)|_H)^{\frac{1}{2}} + |u_x(0) - u_x(t)|_H \right\} \\ &\geq |u_{0x}(x)| - C_1 \left\{ \frac{2T_0}{\mu} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right) \right\}^{\frac{1}{4}} \\ &\quad \times \sqrt{2} \left\{ \frac{2}{\gamma} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right) \right\}^{\frac{1}{4}} \end{aligned}$$

$$\begin{aligned}
& - \sqrt{\frac{2T_0}{\mu}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \\
& \geq |u_{0x}(x)| - \sqrt[4]{T_0} \left(\frac{2C_1}{\sqrt[4]{\gamma\mu}} + \frac{\sqrt{2}\sqrt[4]{T}}{\sqrt{\mu}} \right) \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \\
& =: |u_{0x}(x)| - C_2 \sqrt[4]{T_0} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \\
& \quad \text{for any } (t, x) \in Q(T_0),
\end{aligned}$$

where C_2 is a positive constant depending only on C_1 , γ , μ , and T .

Next, by using (4.3.2) in Lemma 4.5, we have

$$\begin{aligned}
|(\Lambda v)_x(t, x)| & \leq |u_{xx}(t)|_H + |u_x(t)|_H \\
& \leq |u_{xx}(t)|_H + \frac{1}{\sqrt{2}} (|u(t)|_H^2 + |u_{xx}(t)|_H^2)^{\frac{1}{2}} \\
& \leq \left(1 + \frac{1}{\sqrt{2}} \right) |u_{xx}(t)|_H + \left(\sqrt{T_0} \left(\int_0^{T_0} |u_t(t)|_H^2 dt \right)^{\frac{1}{2}} + |u_0|_H \right) \\
& \leq \left(1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{2}{\gamma}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T_0}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \\
& \quad + T_0 \sqrt{\frac{2}{\rho}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T_0}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} + |u_0|_H \\
& \leq C_3 \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 \right)^{\frac{1}{2}} + \sqrt{T_0} C_3 \left(1 + \frac{1}{\delta^4} \right) M + |u_0|_H \\
& \quad \text{for a.e. } (t, x) \in Q(T_0),
\end{aligned}$$

where C_3 is a positive constant. Now, we can choose $\delta > 0$ and $M > 0$ such that

$$|u_{0x}(x)| \geq 2\delta \text{ for } x \in (0, 1), \quad C_3 \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 \right)^{\frac{1}{2}} + |u_0|_H \leq \frac{M}{2}.$$

Moreover, we take $T_0 > 0$ such that

$$C_2 \sqrt[4]{T_0} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa^2 M^2 T}{32\mu} \left(1 + \frac{1}{\delta^4} \right)^2 \right)^{\frac{1}{2}} \leq \delta, \quad \sqrt{T_0} C_3 \left(1 + \frac{1}{\delta^4} \right) \leq \frac{1}{2}.$$

Hence, we infer that

$$\delta \leq |(\Lambda v)_x| \leq M \text{ a.e. on } Q(T_0).$$

Namely, u satisfies (C2).

Next, we show (C3). Thanks to (4.3.2), it follows that

$$|(\Lambda v)_{xx}(t)|_H \leq \sqrt{\frac{2}{\gamma}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 \right)^{\frac{1}{2}} + \frac{\kappa M \sqrt{T_0}}{4\sqrt{\gamma\mu}} \left(1 + \frac{1}{\delta^4} \right) \quad \text{for any } t \in [0, T_0].$$

Now, by choosing $M' > 0$ such that $\sqrt{\frac{2}{\gamma}} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 \right)^{\frac{1}{2}} \leq \frac{M'}{2}$ and $T_0 > 0$, again, such that $\frac{\kappa M \sqrt{T_0}}{4\sqrt{\gamma\mu}} \left(1 + \frac{1}{\delta^4} \right) \leq \frac{M'}{2}$, we have $|(\Lambda v)_{xx}(t)|_H \leq M'$ for a.e. $t \in (0, T_0)$. Thus, (C3) is valid.

As a next step, we prove (C5). By using (4.3.1) and (4.3.2) in Lemma 4.5, we have

$$\begin{aligned} & \int_0^{T_0} |(\Lambda v)_{xxx}(t)|_H^2 dt \\ & \leq \frac{1}{2} \int_0^{T_0} |u_{xx}(t)|_H^2 dt + \frac{1}{2} \int_0^{T_0} |u_{xxxx}(t)|_H^2 dt \\ & \leq \frac{T_0}{\gamma} \left(\frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \beta T_0 \right) + \frac{T_0}{\gamma} \left\{ \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxxx}|_H^2 + \alpha \right\} \\ & \leq C_4 + C_4 T_0 \overline{M}, \end{aligned}$$

where C_4 is a positive constant. Hence, we can choose $\overline{M} > 0$ and $T_0 > 0$, again, such that $C_4 \leq \frac{\overline{M}}{2}$ and $C_4 T_0 \leq \frac{1}{2}$. Immediately, we have

$$\int_0^{T_0} |(\Lambda v)_{xxx}(t)|_H^2 dt \leq \overline{M},$$

namely, (C5) holds. Thus, we have $\Lambda v \in K(T_0, \delta, M, M', \overline{M})$. It means that Λ is the mapping from $K(T_0, \delta, M, M', \overline{M})$ into itself.

Finally, we prove that $\Lambda : K(T_1, \delta, M, M', \overline{M}) \rightarrow K(T_1, \delta, M, M', \overline{M})$ is a contraction mapping for some $T_1 \in (0, T_0]$. Let $v_1, v_2 \in K(T_0, \delta, M, M', \overline{M})$ and $T_1 \in (0, T_0]$. We put $\Lambda v_1 = u_1$, $\Lambda v_2 = u_2$, $v = v_1 - v_2$, and $u = u_1 - u_2$. Since, for each i , u_i satisfies (4.2.1) - (4.2.3), we have

$$\rho u_{tt} + \gamma u_{xxxx} - \mu u_{txx} = (f(\hat{\varepsilon}_1) v_{1x})_x - (f(\hat{\varepsilon}_2) v_{2x})_x \text{ a.e. on } Q(T_0), \quad (4.3.6)$$

$$\frac{\partial^i}{\partial x^i} u(0) = \frac{\partial^i}{\partial x^i} u(1) \text{ a.e. on } (0, T_0) \text{ for } i = 0, 1, 2, 3, \quad (4.3.7)$$

$$u(0, x) = 0, \quad \frac{\partial}{\partial t} u(0, x) = 0 \text{ a.e. on } (0, 1). \quad (4.3.8)$$

By multiplying (4.3.6) by u_t and integrating its both sides, thanks to (4.3.7) we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 \right) + \mu |u_{tx}(t)|_H^2 \\ & = \left((f(\hat{\varepsilon}_1) v_{1x})_x(t) - (f(\hat{\varepsilon}_2) v_{2x})_x(t), u_t(t) \right)_H \\ & = - \left((f(\hat{\varepsilon}_1) v_x)(t), u_{tx}(t) \right)_H - \left(((f(\hat{\varepsilon}_1) - f(\hat{\varepsilon}_2)) v_{2x})(t), u_{tx}(t) \right)_H \\ & \leq \frac{\mu}{2} |u_{tx}(t)|_H^2 + \frac{1}{\mu} \left| (f(\hat{\varepsilon}_1) v_x)(t) \right|_H^2 + \frac{1}{\mu} \left| ((f(\hat{\varepsilon}_1) - f(\hat{\varepsilon}_2)) v_{2x})(t) \right|_H^2 \quad \text{for a.e. } t \in (0, T_0). \end{aligned}$$

On account of (4.3.8) it is easy to see that

$$\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}|_H^2 d\tau$$

$$\begin{aligned}
&\leq \frac{1}{\mu} \left\{ \int_0^t |(f(\hat{\varepsilon}_1)v_x)|_H^2 d\tau + \int_0^t \left| \left((f(\hat{\varepsilon}_1) - f(\hat{\varepsilon}_2))v_{2x} \right) \right|_H^2 d\tau \right\} \\
&=: F_1(t) + F_2(t) \quad \text{for any } t \in [0, T_0].
\end{aligned}$$

First, similarly to (4.3.4), we have

$$F_1(t) \leq \frac{\kappa^2}{16} \left(1 + \frac{1}{\delta^4} \right)^2 |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0].$$

Noting that $f(\hat{\varepsilon}_i) = \frac{\kappa}{4} \left(1 - \frac{1}{(1 + \hat{\varepsilon}_i)^4} \right)$, $\hat{\varepsilon}_i = |v_{ix}| - 1$ on $Q(T_0)$ and $v_i \in K(T_0, \delta, M, M', \overline{M})$ for $i = 1, 2$, we have

$$\begin{aligned}
F_2(t) &\leq \frac{\kappa^2 M^2}{16} \int_0^t \int_0^1 \left| \frac{1}{|v_{1x}|^4} - \frac{1}{|v_{2x}|^4} \right|^2 dx d\tau \\
&\leq \frac{\kappa^2 M^2}{16} \int_0^t \int_0^1 \frac{1}{|v_{1x}|^8 |v_{2x}|^8} \left| |v_{1x}|^4 - |v_{2x}|^4 \right|^2 dx d\tau \\
&\leq \frac{\kappa^2 M^2}{16} \int_0^t \int_0^1 \frac{(|v_{1x}|^2 + |v_{2x}|^2)^2}{|v_{1x}|^8 |v_{2x}|^8} \left| |v_{1x}|^2 - |v_{2x}|^2 \right|^2 dx d\tau \\
&\leq \frac{\kappa^2 M^2}{16} \int_0^t \int_0^1 \frac{(|v_{1x}|^2 + |v_{2x}|^2)^2}{|v_{1x}|^8 |v_{2x}|^8} (|v_{1x}| + |v_{2x}|)^2 |v_x|^2 dx d\tau \\
&\leq \frac{\kappa^2 M^8}{\delta^{16}} \int_0^t |v|_H^2 d\tau \\
&\leq \frac{\kappa^2 M^8}{\delta^{16}} |v|_{W(t)}^2 \quad \text{for } t \in [0, T_0].
\end{aligned} \tag{4.3.9}$$

By these inequalities, we obtain

$$\begin{aligned}
&\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau x}|_H^2 d\tau \\
&\leq \frac{\kappa^2}{\mu} \left\{ \frac{1}{16} \left(1 + \frac{1}{\delta^4} \right)^2 + \frac{M^8}{\delta^{16}} \right\} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0].
\end{aligned} \tag{4.3.10}$$

Moreover, (4.2.4) holds in the case of $F = f(\hat{\varepsilon}_1)v_{1x} - f(\hat{\varepsilon}_2)v_{2x}$, $u_0 = 0$, and $v_0 = 0$. It follows that

$$\begin{aligned}
&\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \\
&\leq \frac{1}{2\mu} \int_0^t |(f(\hat{\varepsilon}_1)v_{1x} - f(\hat{\varepsilon}_2)v_{2x})_{xx}|_H^2 d\tau \\
&\leq \frac{3}{2\mu} \left\{ \int_0^t |((f(\hat{\varepsilon}_1))_{xx} v_{1x}) - ((f(\hat{\varepsilon}_2))_{xx} v_{2x})|_H^2 d\tau \right. \\
&\quad \left. + 4 \int_0^t |((f(\hat{\varepsilon}_1))_x v_{1xx}) - ((f(\hat{\varepsilon}_2))_x v_{2xx})|_H^2 d\tau + \int_0^t |(f(\hat{\varepsilon}_1)v_{1xxx}) - (f(\hat{\varepsilon}_2)v_{2xxx})|_H^2 d\tau \right\} \\
&=: \frac{3}{2\mu} (F_3(t) + F_4(t) + F_5(t)) \quad \text{for any } t \in [0, T_0].
\end{aligned}$$

First, we show that

$$\begin{aligned} F_3(t) &\leq 2 \left(\int_0^t |(f(\hat{\varepsilon}_1))_{xx} v_x|_H^2 d\tau + \int_0^t |((f(\hat{\varepsilon}_1))_{xx} - (f(\hat{\varepsilon}_2))_{xx}) v_{2x}|_H^2 d\tau \right) \\ &=: 2(F_{3,1}(t) + F_{3,2}(t)) \end{aligned} \quad \text{for any } t \in [0, T_0].$$

By using $(f(\hat{\varepsilon}_i))_{xx} = \kappa \left(\frac{1}{|v_{ix}|^4} - \frac{6|v_{ix} \cdot v_{ixx}|^2}{|v_{ix}|^8} + \frac{v_{ix} \cdot v_{ixxx}}{|v_{ix}|^6} \right)$ on $Q(T_0)$,

$v_i \in K(T_0, \delta, M, M', \overline{M})$ for $i = 1, 2$, and (4.3.5), we have

$$\begin{aligned} F_{3,1}(t) &\leq 3\kappa^2 \left\{ \frac{1}{\delta^8} \int_0^t |v_x|_H^2 d\tau + \frac{36}{\delta^{12}} \int_0^t \int_0^1 |v_{1xx}|^4 |v_x|^2 dx d\tau + \frac{1}{\delta^{10}} \int_0^t \int_0^1 |v_{1xxx}|^2 |v_x|^2 dx d\tau \right\} \\ &\leq 3\kappa^2 \left\{ \frac{1}{\delta^8} |v|_{L^2(0,t;H^3(0,1)^2)}^2 \right. \\ &\quad \left. + \frac{36}{\delta^{12}} |v_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \int_0^t \int_0^1 (|v_{1xx}|_H^2 + |v_{1xxx}|_H^2) |v_{1xx}|^2 dx d\tau \right. \\ &\quad \left. + \frac{1}{\delta^{10}} |v_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \int_0^t |v_{1xxx}|_H^2 d\tau \right\} \\ &\leq 3\kappa^2 \left\{ \frac{1}{\delta^8} |v|_{L^2(0,t;H^3(0,1)^2)}^2 + \frac{72(M')^2}{\delta^{12}} \left\{ (M')^2 T + \overline{M} \right\} |v_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \right. \\ &\quad \left. + \frac{\overline{M}}{\delta^{10}} |v_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \right\} \\ &\leq 3\kappa^2 \left\{ \frac{1}{\delta^8} + \frac{72(M')^2}{\delta^{12}} \left\{ (M')^2 T + \overline{M} \right\} + \frac{\overline{M}}{\delta^{10}} \right\} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0]. \end{aligned}$$

Easily, we get

$$\begin{aligned} F_{3,2}(t) &\leq \kappa^2 \int_0^t \int_0^1 \left\{ \frac{1}{|v_{1x}|^4 |v_{2x}|^4} \left| |v_{1x}|^4 - |v_{2x}|^4 \right| + 6 \left| \frac{|v_{1x} \cdot v_{1xx}|^2}{|v_{1x}|^8} - \frac{|v_{2x} \cdot v_{2xx}|^2}{|v_{2x}|^8} \right| \right. \\ &\quad \left. + \frac{1}{|v_{1x}|^6 |v_{2x}|^6} \left| |v_{2x}|^6 (v_{1x} \cdot v_{1xxx}) - |v_{1x}|^6 (v_{2x} \cdot v_{2xxx}) \right| \right\}^2 |v_{2x}|^2 dx d\tau \\ &\leq 3\kappa^2 M^2 \left\{ \int_0^t \int_0^1 \frac{1}{|v_{1x}|^8 |v_{2x}|^8} \left| |v_{1x}|^4 - |v_{2x}|^4 \right|^2 dx d\tau \right. \\ &\quad \left. + \int_0^t \int_0^1 \frac{36}{|v_{1x}|^{16} |v_{2x}|^{16}} \left| |v_{2x}|^8 |v_{1x} \cdot v_{1xx}|^2 - |v_{1x}|^8 |v_{2x} \cdot v_{2xx}|^2 \right|^2 dx d\tau \right. \\ &\quad \left. + \frac{1}{|v_{1x}|^{12} |v_{2x}|^{12}} \left| |v_{2x}|^6 (v_{1x} \cdot v_{1xxx}) - |v_{1x}|^6 (v_{2x} \cdot v_{2xxx}) \right|^2 dx d\tau \right\} \\ &=: 3\kappa^2 M^2 (F_{3,2,1}(t) + F_{3,2,2}(t) + F_{3,2,3}(t)) \quad \text{for any } t \in [0, T_0]. \end{aligned}$$

Similarly to (4.3.9), we have

$$F_{3,2,1}(t) \leq \frac{16M^6}{\delta^{16}} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0].$$

By using (4.3.5), we observe that

$$\begin{aligned}
& F_{3,2,2}(t) \\
&= 36 \int_0^t \int_0^1 \frac{1}{|v_{1x}|^{16} |v_{2x}|^{16}} \left| |v_{2x}|^8 |v_{1x} \cdot v_{1xx}|^2 - |v_{2x}|^8 (v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx}) \right. \\
&\quad \left. + |v_{2x}|^8 (v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx}) - |v_{1x}|^8 (v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx}) \right. \\
&\quad \left. + |v_{1x}|^8 (v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx}) - |v_{1x}|^8 |v_{2x} \cdot v_{2xx}|^2 \right|^2 dx dt \\
&= 36 \int_0^t \int_0^1 \frac{1}{|v_{1x}|^{16} |v_{2x}|^{16}} \\
&\quad \times \left| |v_{2x}|^8 (v_{1x} \cdot v_{1xx}) \{ (v_{1x} \cdot v_{1xx}) - (v_{1x} \cdot v_{2xx}) + (v_{1x} \cdot v_{2xx}) - (v_{2x} \cdot v_{2xx}) \} \right. \\
&\quad \left. + (|v_{2x}|^8 - |v_{1x}|^8) (v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx}) \right. \\
&\quad \left. + |v_{1x}|^8 (v_{2x} \cdot v_{2xx}) \right. \\
&\quad \left. \times \{ (v_{1x} \cdot v_{1xx}) - (v_{1x} \cdot v_{2xx}) + (v_{1x} \cdot v_{2xx}) - (v_{2x} \cdot v_{2xx}) \} \right|^2 dx d\tau \\
&= 36 \int_0^t \int_0^1 \left| \frac{1}{|v_{1x}|^8} (v_{1x} \cdot v_{1xx}) \{ (v_{1x} \cdot v_{xx}) + (v_x \cdot v_{2xx}) \} \right. \\
&\quad \left. + \frac{(v_{1x} \cdot v_{1xx})(v_{2x} \cdot v_{2xx})}{|v_{1x}|^8 |v_{2x}|^8} (|v_{2x}|^8 - |v_{1x}|^8) \right. \\
&\quad \left. + \frac{1}{|v_{2x}|^8} (v_{2x} \cdot v_{2xx}) \{ (v_{1x} \cdot v_{xx}) + (v_x \cdot v_{2xx}) \} \right|^2 dx d\tau \\
&\leq 36 \int_0^t \int_0^1 \left| \frac{1}{|v_{1x}|^8} (|v_{1x}|^2 |v_{1xx}| |v_{xx}| + |v_{1x}| |v_{1xx}| |v_{2xx}| |v_x|) \right. \\
&\quad \left. + \frac{|v_{1x}| |v_{1xx}| |v_{2x}| |v_{2xx}|}{|v_{1x}|^8 |v_{2x}|^8} (|v_{2x}|^4 + |v_{1x}|^4) (|v_{2x}|^2 + |v_{1x}|^2) (|v_{2x}| + |v_{1x}|) |v_x| \right. \\
&\quad \left. + \frac{1}{|v_{2x}|^8} (|v_{2x}| |v_{2xx}| |v_{1x}| |v_{xx}| + |v_{2x}| |v_{2xx}|^2 |v_x|) \right|^2 dx d\tau \\
&\leq 108 \left(\frac{2M^2}{\delta^{14}} \int_0^t \int_0^1 |v_{1xx}|^2 |v_{xx}|^2 dx d\tau + \frac{2}{\delta^{14}} \int_0^t \int_0^1 |v_{1xx}|^2 |v_{2xx}|^2 |v_x|^2 dx d\tau \right. \\
&\quad \left. + \frac{64M^{14}}{\delta^{28}} \int_0^t \int_0^1 |v_{1xx}|^2 |v_{2xx}|^2 |v_x|^2 dx d\tau + \frac{2M^2}{\delta^{14}} \int_0^t \int_0^1 |v_{2xx}|^2 |v_{xx}|^2 dx d\tau \right. \\
&\quad \left. + \frac{2}{\delta^{14}} \int_0^t \int_0^1 |v_{2xx}|^4 |v_x|^2 dx d\tau \right) \\
&\leq C_5 \left(\int_0^t \int_0^1 |v_{1xx}|^2 |v_{xx}|^2 dx d\tau + \int_0^t \int_0^1 |v_{1xx}|^2 |v_{2xx}|^2 |v_x|^2 dx d\tau \right. \\
&\quad \left. + \int_0^t \int_0^1 |v_{2xx}|^2 |v_{xx}|^2 dx d\tau + \int_0^t \int_0^1 |v_{2xx}|^4 |v_x|^2 dx d\tau \right) \quad \text{for any } t \in [0, T_0],
\end{aligned}$$

where C_5 is a positive constant. Moreover, we see that

$$\begin{aligned} F_{3,2,2}(t) &\leq 4(M')^2 C_5 \left(\int_0^t (|v_{xx}|_H^2 + |v_{xxx}|_H^2) d\tau \right. \\ &\quad \left. + |v_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \int_0^t (|v_{1xx}|_H^2 + |v_{1xxx}|_H^2) d\tau \right) \\ &\leq C_6 |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0], \end{aligned}$$

where C_6 is a positive constant. Similarly, we can get

$$F_{3,2,3}(t) \leq C_7 |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0],$$

where C_7 is a positive constant.

Thus, we obtain

$$F_{3,2}(t) \leq C_8 |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0],$$

where C_8 is a positive constant. Hence, for some positive constant C_9 , we have

$$F_3(t) \leq C_9 |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0].$$

Similarly, we obtain

$$F_4(t) \leq C_{10} |v|_{W(t)}^2, \quad F_5(t) \leq C_{10} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0] \quad \text{for any } t \in [0, T_0],$$

where C_{10} is a positive constant. Hence, we obtain

$$\frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxxx}(t)|_H^2 + \frac{\mu}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \leq C_{11} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0], \quad (4.3.11)$$

where C_{11} is a positive constant. By using (4.3.10) and (4.3.11), it is easy to see that

$$|u(t)|_{H^3(0,1)^2}^2 \leq C_{12} |v|_{W(t)}^2, \quad |u|_{L^2(0,t;H^3(0,1)^2)}^2 \leq C_{12} t |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0], \quad (4.3.12)$$

where C_{12} is a positive constant. Here, we note the following inequality:

$$|z| \leq \left(\sqrt{2} + 1 \right) \left(|z|_H^{\frac{1}{2}} |z_x|_H^{\frac{1}{2}} + |z|_H \right) \quad \text{on } (0, 1) \quad \text{for } z \in H^1(0, 1)^2.$$

By using this inequality, (4.3.10) and (4.3.12), we have

$$\begin{aligned} |u_x(t, x)| &\leq \left(\sqrt{2} + 1 \right) \left(|u_x(t)|_H^{\frac{1}{2}} |u_{xx}(t)|_H^{\frac{1}{2}} + |u_x(t)|_H \right) \\ &\leq C_{13} |v|_{W(t)} \quad \text{for } (t, x) \in \overline{Q(T_0)}, \end{aligned}$$

where C_{13} a positive constant. Hence, we obtain

$$|u_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \leq C_{13}^2 t^{\frac{1}{2}} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0]. \quad (4.3.13)$$

Since we define the norm of W by $|z|_{W(t)} = \left(|z|_{L^2(0,t;H^3(0,1)^2)}^2 + |z_x|_{L^\infty(0,t;L^\infty(0,1)^2)}^2 \right)^{\frac{1}{2}}$, from (4.3.12) and (4.3.13) it follows that

$$|u|_{W(t)}^2 \leq \left(C_{12}\sqrt{T} + C_{13} \right) \sqrt{t} |v|_{W(t)}^2 \quad \text{for any } t \in [0, T_0].$$

Therefore, by choosing $T_1 \in (0, T_0]$ such that $T_1 \leq \frac{1}{(C_{12}\sqrt{T} + C_{13})^2}$, we see that

$\Lambda : K(T_1, \delta, M, M', \overline{M}) \rightarrow K(T_1, \delta, M, M', \overline{M})$ is contractive. Hence, Lemma 4.4 is proved. \square

Proof of Theorem 4.2. By Lemma 4.4, there exists $T_1 > 0$ such that $\Lambda : K(T_1, \delta, M, M', \overline{M}) \rightarrow K(T_1, \delta, M, M', \overline{M})$ is contractive. Therefore, the Banach fixed point theorem implies the existence of one and only one $u \in K(T_1, \delta, M, M', \overline{M})$ such that $\Lambda u = u$. Thus, P_μ has a unique strong solution u on $Q(T_1)$.

From now on in order to show that we can extend the strong solution to a function on $Q(T)$, we give uniform estimates for the strong solution with respect to $t \in [0, T]$.

First, we multiply (1.3.11) by u_t and integrate the both sides with respect to x , and we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \frac{\kappa}{8} \int_0^1 \left(|u_x(t, x)|^2 + \frac{1}{|u_x(t, x)|^2} \right) dx \right) + \mu |u_{tx}(t)|_H^2 \\ & = 0 \end{aligned} \quad \text{for any } t \in (0, T_1),$$

and by integrating it with respect to t , we obtain

$$\begin{aligned} & \frac{\rho}{2} |u_t(t)|_H^2 + \frac{\gamma}{2} |u_{xx}(t)|_H^2 + \mu \int_0^t |u_{\tau x}|_H^2 d\tau + \frac{\kappa}{8} \int_0^1 \left(|u_x(t, x)|^2 + \frac{1}{|u_x(t, x)|^2} \right) dx \\ & = \frac{\rho}{2} |v_0|_H^2 + \frac{\gamma}{2} |u_{0xx}|_H^2 + \frac{\kappa}{8} \int_0^1 \left(|u_x(0, x)|^2 + \frac{1}{|u_x(0, x)|^2} \right) dx \\ & =: \tilde{C}_1 \end{aligned} \quad \text{for any } t \in [0, T_1]. \quad (4.3.14)$$

By using Lemma 4.3 and (4.3.14), there exists a positive constant δ such that

$$|u_x(T_1, x)| \geq \delta \quad \text{for any } x \in [0, 1]. \quad (4.3.15)$$

Moreover, from (4.2.4) with $F = f(\varepsilon)u_x$ it follows that

$$\begin{aligned} & \frac{\rho}{2} |u_{txx}(t)|_H^2 + \frac{\gamma}{2} |u_{xxx}(t)|_H^2 + \frac{\gamma}{2} \int_0^t |u_{\tau xxx}|_H^2 d\tau \\ & \leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxx}|_H^2 + \frac{1}{\mu} \int_0^t |(f(\varepsilon)u_x)_{xx}|_H^2 d\tau \\ & \leq \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxx}|_H^2 \\ & \quad + \frac{3}{\mu} \left(\int_0^t |f(\varepsilon)u_{xxx}|_H^2 d\tau + 4 \int_0^t |(f(\varepsilon))_x u_{xx}|_H^2 d\tau + \int_0^t |(f(\varepsilon))_{xx} u_x|_H^2 d\tau \right) \\ & =: \frac{\rho}{2} |v_{0xx}|_H^2 + \frac{\gamma}{2} |u_{0xxx}|_H^2 + \frac{3}{\mu} (G_1(t) + G_2(t) + G_3(t)) \end{aligned} \quad \text{for any } t \in [0, T_1].$$

We provide estimates for G_1 , G_2 , and G_3 as follows. First, by using $f(\varepsilon) = \frac{\kappa}{4} \left(1 - \frac{1}{(1+\varepsilon)^4}\right)$ and $\varepsilon = |u_x| - 1$ on $Q(T_1)$, we have

$$\begin{aligned} G_1(t) &\leq \frac{\kappa^2}{16} \left(1 + \frac{1}{\delta^4}\right)^2 \int_0^t |u_{xxx}|^2 d\tau \\ &\leq \frac{\kappa^2}{32} \left(1 + \frac{1}{\delta^4}\right)^2 \int_0^t (|u_{xx}|_H^2 + |u_{xxxx}|_H^2) d\tau \\ &\leq \frac{\kappa^2 \tilde{C}_1}{16} \left(1 + \frac{1}{\delta^4}\right)^2 t + \frac{\kappa^2}{32} \left(1 + \frac{1}{\delta^4}\right)^2 \int_0^t |u_{xxxx}|_H^2 d\tau \quad \text{for any } t \in [0, T_1]. \end{aligned}$$

Next, since $(f(\varepsilon))_x = \kappa \frac{u_x \cdot u_{xx}}{|u_x|^6}$ on $(0, T_1) \times (0, 1)$ and $u_{xx}(t)$ is continuous on $[0, 1]$ for any $t \in [0, T_1]$, by (4.3.14), we see that

$$\begin{aligned} G_2(t) &\leq \frac{4\kappa^2}{\delta^4} \int_0^t \int_0^1 |u_{xx}|^4 dx d\tau \\ &\leq \frac{4\kappa^2}{\delta^4} \int_0^t \left(\max_{x \in [0, 1]} |u_{xx}(\tau, x)| \right)^2 |u_{xx}(\tau)|_H^2 d\tau \quad \text{for any } t \in [0, T_1]. \end{aligned} \quad (4.3.16)$$

Now, by (4.3.5), the following inequality holds:

$$\begin{aligned} |u_{xx}(t, x)| &\leq |u_{xx}(t)|_H + |u_{xxx}(t)|_H \\ &\leq |u_{xx}(t)|_H + \frac{1}{\sqrt{2}} \left(|u_{xx}(t)|_H + |u_{xxxx}(t)|_H \right) \\ &= \left(1 + \frac{1}{\sqrt{2}}\right) |u_{xx}(t)|_H + \frac{1}{\sqrt{2}} |u_{xxxx}(t)|_H \quad \text{for any } (t, x) \in Q(T_1). \end{aligned} \quad (4.3.17)$$

Thanks to (4.3.16) and (4.3.17), easily, we get

$$\begin{aligned} G_2(t) &\leq \frac{8\kappa^2 \tilde{C}_1}{\gamma \delta^4} \int_0^t \left\{ \left(1 + \frac{1}{\sqrt{2}}\right)^2 |u_{xx}|_H^2 + \frac{1}{2} |u_{xxxx}|_H^2 \right\} d\tau \\ &\leq \frac{32\kappa^2 \tilde{C}_1^2}{\gamma^2 \delta^4} \left(1 + \frac{1}{\sqrt{2}}\right)^2 t + \frac{8\kappa^2 \tilde{C}_1}{\gamma \delta^4} \int_0^t |u_{xxxx}|_H^2 d\tau \quad \text{for any } t \in [0, T_1]. \end{aligned}$$

Also, we note that $(f(\varepsilon))_{xx} = \frac{\kappa}{|u_x|^2} \left(\frac{1}{|u_x|^2} - \frac{6|u_x \cdot u_{xx}|^2}{|u_x|^6} + \frac{u_x \cdot u_{xxx}}{|u_x|^4} \right)$ on $Q(T_1)$. Similarly, it holds that

$$\begin{aligned} G_3(t) &\leq 3\kappa^2 \int_0^t \int_0^1 \left(\frac{1}{|u_x|^4} + \frac{36|u_{xx}|^4}{|u_x|^8} + \frac{|u_{xxx}|^2}{|u_x|^6} \right) dx d\tau \\ &\leq 3\kappa^2 \left(\frac{1}{\delta^4} t + \frac{36}{\delta^8} \int_0^t \int_0^1 |u_{xx}|^4 dx d\tau + \frac{1}{\delta^6} \int_0^t |u_{xxx}|_H^2 d\tau \right) \\ &\leq R_1 t + R_2 \int_0^t |u_{xxxx}|_H^2 d\tau \quad \text{for any } t \in [0, T_1], \end{aligned}$$

where R_1 and R_2 are positive constants independent of T_1 . Thus, we have

$$\begin{aligned} & \frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \frac{\gamma}{2}\int_0^t |u_{\tau xxx}|_H^2 d\tau \\ & \leq \frac{\rho}{2}|v_{0xx}|_H^2 + \frac{\gamma}{2}|u_{0xxxx}|_H^2 + \hat{C}_1 t + \hat{C}_2 \int_0^t |u_{xxxx}|_H^2 d\tau \quad \text{for any } t \in [0, T_1], \end{aligned} \quad (4.3.18)$$

where \hat{C}_1 and \hat{C}_2 are positive constants independent of T_1 . By applying the Gronwall inequality to (4.3.18), there exists a positive constant \tilde{C}_1 independent of T_1 such that

$$\frac{\rho}{2}|u_{txx}(t)|_H^2 + \frac{\gamma}{2}|u_{xxxx}(t)|_H^2 + \frac{\gamma}{2}\int_0^t |u_{\tau xxx}|_H^2 d\tau \leq \tilde{C}_1 \quad \text{for any } t \in [0, T_1]. \quad (4.3.19)$$

From (4.3.15) and (4.3.19), we can extend u to a strong solution on $[0, T_1 + T_2]$ for some positive constant T_2 which is independent of T_1 . Hence, we can extend it to a solution of P_μ on $[0, T]$ recursively. Thus, Theorem 4.2 has been proved. \square

4.4 Existence of weak solutions

In this section we prove existence of a weak solution for P_μ on $Q(T)$. We suppose all assumptions of Theorem 4.1.

Proof of the existence of weak solutions: As in the proof of Lemma 4.2 in Section 4.2, we can take sequences $\{u_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V_2$ and $\{v_{0n}\}_{n \in \mathbb{Z}_{>0}} \subset V_1$ such that

$$|u_{0nx}| \geq \hat{\delta} \text{ on } [0, 1] \text{ for } n \in \mathbb{Z}_{>0}, u_{0n} \rightarrow u_0 \text{ in } V_1 \text{ and } v_{0n} \rightarrow v_0 \text{ in } H \text{ as } n \rightarrow \infty, \quad (4.4.1)$$

where $\hat{\delta}$ is a positive constant. Now, for all $n \in \mathbb{Z}_{>0}$, $u_{0n} \in V_2$ and $v_{0n} \in V_1$, the pair of u_{0n} and v_{0n} satisfies the assumption of Theorem 4.2. Accordingly, for all $n \in \mathbb{Z}_{>0}$, Theorem 4.2 guarantees the existence of a strong solution u_n of P_μ on $Q(T)$ with the initial functions u_{0n} and v_{0n} . Obviously, by (4.3.14) there exists a positive constant C_0 independent of n such that

$$\begin{aligned} & \frac{\rho}{2}|u_{nt}(t)|_H^2 + \frac{\gamma}{2}|u_{nxx}(t)|_H^2 + \mu \int_0^t |u_{n\tau x}(\tau)|_H^2 d\tau + \frac{\kappa}{8} \left(|u_{nx}(t)|_H^2 + \int_0^1 \frac{1}{|u_{nx}(t, x)|^2} dx \right) \\ & \leq \frac{\rho}{2}|v_{0n}|_H^2 + \frac{\gamma}{2}|u_{0nxx}|_H^2 + \frac{\kappa}{8}|u_{0nx}|_H^2 + \frac{\kappa}{8} \int_0^1 \frac{1}{|u_{0nx}(x)|^2} dx \\ & \leq C_0 \quad \text{for } t \in [0, T] \text{ and } n \in \mathbb{Z}_{>0}. \end{aligned} \quad (4.4.2)$$

This estimate together with Lemma 4.3 implies that

$$|u_{nx}| \geq \sqrt{\frac{C_0}{\gamma}} \exp\left(-\frac{16C_0^2}{\gamma\kappa}\right) \quad \text{on } Q(T) \text{ for } n \geq N_1.$$

Hence, these estimates guarantee existence of $u \in L^2(0, T; V_1) \cap W^{1,2}(0, T; X) \cap C(\overline{Q(T)})$ and a subsequence $\{n_j\}_{j \in \mathbb{Z}_{>0}} \subset \{n\}_{n \in \mathbb{Z}_{>0}}$ such that

$$u_{n_j} \rightarrow u \text{ weakly in } L^2(0, T; V_1), \quad u_{n_j} \rightarrow u \text{ weakly in } W^{1,2}(0, T; X),$$

$$u_{n_j x} \rightarrow u_x \text{ in } C(\overline{Q(T)}), \quad u_{n_j}(t) \rightarrow u(t) \text{ in } H \text{ as } j \rightarrow \infty \text{ for any } t \in [0, T]. \quad (4.4.3)$$

Now, for all $n \in \mathbb{Z}_{>0}$, u_n also satisfies the definition of the weak solution for P_μ on $Q(T)$, namely, for all $n_j \geq N_1$, u_{n_j} satisfying $u_{n_j}(0) = u_{0n_j}$, $|u_{n_j x}| > 0$ on $\overline{Q(T)}$ and

$$\begin{aligned} & -\rho \int_{Q(T)} u_{n_j t} \cdot \eta_t dx dt + \gamma \int_{Q(T)} u_{n_j x x} \cdot \eta_{x x} dx dt \\ & + \int_{Q(T)} f(\varepsilon_{n_j}) u_{n_j x} \cdot \eta_x dx dt + \mu \int_{Q(T)} u_{n_j t x} \cdot \eta_x dx dt = \int_0^1 v_{0n_j} \cdot \eta(0) dx \end{aligned} \quad (4.4.4)$$

for $\eta \in W^{1,2}(0, T; H) \cap L^2(0, T; V_1)$ with $\eta(T) = 0$ a.e. on $(0, 1)$.

Thus, by using (4.4.1), (4.4.2), (4.4.3) and letting $j \rightarrow \infty$ in (4.4.4), we infer that u satisfies the required conditions in Definition 4.2. Hence, the existence of weak solutions to P_μ on $Q(T)$ has been proved. \square

Chapter 5

Future work

In this chapter, we consider mathematical and numerical issues for each model given in thesis.

5.1 The ordinary differential equation model

As mentioned in Sections 2.6 and 2.7, our model (OP) has a periodic solution in time if initial functions are symmetric in some sense. However, our simulations indicate unstable behavior as in Figures 2.4, 2.6, 2.8 and 2.9, even if we applied the structure preserving method. More precisely, such unstable behavior is observed for any initial data, and when $N = 1$, the numerical solution seems to be stable. For these numerical results, we give a list of issues as follows:

- Clarify conditions for initial data and the dimension N , such that the numerical solution by our scheme is unstable.
- Develop a new scheme to obtain stable numerical solutions.

In Chapter 2, we obtain the error estimate for the scheme (NS) in Theorem 2.3 as follows:

$$|X(t) - X^K(t)| \leq C \left| \frac{T}{K} \right|^\alpha \text{ for } 0 \leq t \leq T \text{ and } K \geq K_0,$$

where $\alpha = 1$, X is a solution of (OP) on $[0, T]$ and X^K is an approximate solution by (NS), and C is a positive constant. On the structure preserving numerical method, $\alpha = 2$ is often observed as a convergence rate in previous results. Thus, we give a future task:

- Improve the convergence rate for (NS).

5.2 The partial differential equation model P_0

In Chapter 3, we establish existence and uniqueness of a solution to P_0 for the stress function f which is Lipschitz continuous, monotone increasing and $f(0) = 0$. From the observation to the proof of the existence, we guess that the conditions for f can be relaxed, since it is essential that the following energy is conserved:

$$\frac{\rho}{2} \int_0^1 |u_t|^2 dx + \frac{\gamma}{2} \int_0^1 |u_{xx}|^2 dx + \frac{1}{2} \int_0^1 \widehat{g}(|u_x|^2) dx = \text{constant on } [0, T].$$

Accordingly, P_0 may have a periodic solution in time. From this argument, we would like to:

- Relax conditions for stress functions such that P_0 has a unique solution.
- Prove existence of a periodic solution in time.

5.3 The partial differential equation model P_μ

For $\mu > 0$ by applying the energy estimate (4.3.14) we prove existence of a weak solution in Chapter 4. We suppose that we can prove existence of a weak solution of P_0 with stress function having the singularity. For uniqueness we guess that the standard method does not work well, and then the proof is one of mathematical challenges.

In Chapter 4, we give the stress function as the exact form, $f(\varepsilon) = \frac{\kappa}{4} \left(1 - \frac{1}{(1+\varepsilon)^4} \right)$ for $\varepsilon > -1$. As mentioned in Chapter 4, the other examples of the stress functions are studied in [15, 25, 30]. Accordingly, it is necessary to show that P_μ has a solution for these stress functions. Thus, we have the following plans as future work:

- Let u_μ be a solution to P_μ with the stress function having the singularity for $\mu > 0$. Prove convergence of the sequence $\{u_\mu\}$ to a weak or strong solution of P_0 .
- Find a class of the stress function having the singularity has a unique solution.

For P_μ , we add the viscosity term. By existence of this term, the energy decays as in (4.4.2). Accordingly, we guess that the solution $u(t)$ converges to the following stationary problem P_∞ as $t \rightarrow \infty$.

$$\begin{aligned} \gamma \frac{\partial^4 u_\infty}{\partial x^4} - \frac{\partial}{\partial x} \left(f(\varepsilon_\infty) \frac{\partial u_\infty}{\partial x} \right) &= 0, \varepsilon_\infty = \left| \frac{\partial u_\infty}{\partial x} \right| - 1 \quad \text{on } (0, 1), \\ \frac{\partial^i}{\partial x^i} u_\infty(0) &= \frac{\partial^i}{\partial x^i} u_\infty(1) \quad \text{for any } i = 0, 1, 2, 3. \end{aligned}$$

Also, one of important issue on this research is to clarify the role of the fourth derivative term γu_{xxxx} . For this aim we suppose that analysis of the stationary solution is effective. Hence, we can give the following list of issues on large time behavior of solutions.

- Prove convergence of $\{u(t) \mid t \geq 0\}$ to the solution u_∞ of P_∞ as $t \rightarrow \infty$.
- Show the relationship of the geometric property of the stationary solution to the value of γ .

5.4 Far-future aims

As mentioned in Chapter 1, this research is motivated by analysis of rapid rotational motion of shape memory alloy rings whose part touches hot water. For the analysis it is necessary to make a mathematical model describing the role of pulleys as in Figure 1.3. On the other hand, satisfactory results on the obstacle problem to elastic materials are not shown, yet. Thus, we have the following far-future plans: Our ideas for the modeling is to regard the role of the pulleys an obstacle problem.

- Construct a mathematical model such that we can solve the obstacle problem. Now, we consider that a key of the construction is application of the stress function having the singularity.
- Show the rapid rotational motion in numerical simulations, and it as theoretical result. For these aims we would like to propose a mathematical model and a numerical scheme.

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