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# Evolution equations generated by subdifferential operators with time-dependent domains

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## Abstract

We will treat a phenomenon occurring on a region which varies with respect to time  $t$ . This problem yields an evolution equation with a time-dependent domain. In this paper we show that the equation has a global bounded solution under appropriate conditions.

Key words: subdifferential operator, time-dependent domain

## 1 Introduction

Non-autonomous evolution equations which are generated by subdifferential operators  $\partial\varphi^t$  with time-dependent domains are important. Ôtani<sup>11)</sup>, Kenmochi<sup>5)</sup> and Yamada<sup>15)</sup> studied such type of equations and obtained many results.

Now, if we are going to treat a phenomenon occurring on a region which varies with respect to time  $t$ , then we encounter an evolution equation with a time-dependent domain. To describe such a problem concretely, we give a moving boundary problem. Let  $\Omega(t) \subset \mathbb{R}^N (t \in \mathbb{R})$  be bounded domains included in a fixed ball  $B$  of  $\mathbb{R}^N$  with sufficiently smooth boundaries  $\partial\Omega(t) (t \in \mathbb{R})$ . As an example, we illustrate the Navier-Stokes equation:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u + f(t), & \operatorname{div} u = 0 & \text{in } \hat{\Omega}_\tau, \\ u|_{\partial\Omega(t)} = 0, & \text{for any } t \in (\tau, \infty), & u|_\tau = u_\tau & \text{in } \Omega(\tau), \end{cases} \quad (1.1)$$

where  $\hat{\Omega}_\tau = \cup_{\tau < t < \infty} (\Omega(t) \times \{t\})$ ,  $\tau \in \mathbb{R}$ .

Concerning moving boundary problems, there are many works. Among them we refer to Fujita-Sauer<sup>3)</sup>, Ôtani-Yamada<sup>12)</sup>, Inoue-Ôtani<sup>4)</sup>. The author also studied in Ôeda<sup>6),7),9),10)</sup>. The purpose of the present paper is to discuss several equations in time-dependent regions  $\hat{\Omega}_\tau$  and to get results on them by using a common framework. Applications will be given in §4.

## 2 Formulation of evolution equations and results

Let  $H$  and  $V$  be two Hilbert spaces with  $V \subset H$  and let  $V'$  be the dual space of  $V$ . We identify  $H$  with a subspace of  $V'$ . We note  $V \subset H \subset V'$ . For  $u \in V$  we usually use the norm  $\|u\|_V$ , but when we regard as  $u \in H$  then we consider the norm  $\|u\|_H$ .

Let  $t \in \mathbb{R}$  be a parameter and  $\varphi^t$  be a proper lower semicontinuous convex function on  $H$  with an effective domain  $D(\varphi^t) = \{u \in H; \varphi^t(u) < \infty\}$ . For each  $u \in D(\varphi^t)$ , the subdifferential  $\partial\varphi^t$  of  $\varphi^t$  at  $u$  is defined as  $\partial\varphi^t(u) = \{w \in H; \varphi^t(v) - \varphi^t(u) \geq (w, v - u) \text{ for all } v \in H\}$ . The

domain of the operator  $\partial\varphi^t$  is defined by  $D(\partial\varphi^t) = \{u \in D(\varphi^t); \partial\varphi^t(u) \neq \emptyset\}$ . We note  $D(\varphi^t)$  and  $D(\partial\varphi^t)$  are time-dependent domains in general.

We assume:

- (A1)  $V$  is dense in  $H$  and the injection from  $V$  to  $H$  is compact. Moreover  $\|u\|_H \leq \|u\|_V$ .
- (A2)  $D(\varphi^t) \subset V$  and  $\varphi^t(u) \geq C_1\|u\|_V^2 \geq C_1'\varphi^t(u)$  for  $u \in D(\varphi^t)$ ,  $C_1, C_1' > 0$ .
- (A3)  $(g, u) = k\varphi^t(u)$  for  $u \in D(\partial\varphi^t)$  and  $g \in \partial\varphi^t(u)$  (Green's formula type equality), where  $k$  is a positive constant.
- (A4) There exist positive constants  $C_2$  and  $C_3$  and  $\tau_0 > 0$  such that for every  $t_0 \in (-\infty, \infty)$  and  $u_0 \in D(\varphi^{t_0})$ , there exist numbers  $s^*(t_0), t^*(t_0)$  and an  $H$ -valued absolutely continuous function  $v(t)$  on a closed interval  $I(t_0) \equiv [\max\{t_0 - \tau_0, s^*\}, \min\{t_0 + \tau_0, t^*\}]$  satisfying

$$\|v(t) - u_0\|_H \leq C_2 \cdot |t - t_0| \cdot \varphi^{t_0}(u_0)^{1/2} \quad \text{for any } t \in I(t_0), \quad (2.1)$$

$$\varphi^t(v(t)) \leq \varphi^{t_0}(u_0) + C_3 \cdot |t - t_0| \cdot \varphi^{t_0}(u_0) \quad \text{for any } t \in I(t_0). \quad (2.2)$$

- (A5) If  $g \in \partial\varphi^t(u)$ ,  $h \in \partial\varphi^t(v)$ , then  $\alpha g + \beta h \in \partial\varphi^t(\alpha u + \beta v)$  for  $\alpha, \beta \in \mathbb{R}$ .

**Remark 1.** From (A1) and (A2), it holds that  $\varphi^t(u) \geq C_1\|u\|_H^2$  for  $u \in D(\varphi^t)$ , that is, Poincaré's type inequality is ensured. We know  $\partial\varphi^t$  is a monotone operator in  $H$ ,  $D(\partial\varphi^t)$  is dense in  $D(\varphi^t)$  under  $H$  norm, and  $\overline{D(\partial\varphi^t)}^H = \overline{D(\varphi^t)}^H$ .

Let  $B^t$  be a bilinear operator mapping  $V \times V$  into  $V'$  and  $D(\partial\varphi^t) \times D(\partial\varphi^t)$  into  $H$  such that

$$(B0) \quad (B^t(u, v), w) = -(B^t(u, w), v) \quad \text{for any } u, v, w \in V, \quad (2.3)$$

$$(B1) \quad \|B^t(u, v)\| \leq C_4\|u\|_H^{\theta_4} \cdot \|u\|_V^{1-\theta_4} \cdot \|v\|_V^{1-\theta_4} \cdot \|g\|_H^{\theta_4} \quad (2.4)$$

for any  $u \in V$ ,  $v \in D(\partial\varphi^t)$  and  $g \in \partial\varphi^t(v)$ ,

$$(B2) \quad \|B^t(u, v)\| + \|B^t(v, u)\| \leq C_5\|u\|_V \cdot \|v\|_V^{1-\theta_5} \cdot \|g\|_H^{\theta_5} \quad (2.5)$$

for any  $u \in V$ ,  $v \in D(\partial\varphi^t)$  and  $g \in \partial\varphi^t(v)$ ,

$$(B3) \quad |(B^t(u, v), w)| \leq C_6\|u\|_H^{\theta_6} \cdot \|u\|_V^{1-\theta_6} \cdot \|v\|_V \cdot \|w\|_V^{\theta_6} \cdot \|w\|_H^{1-\theta_6} \quad (2.6)$$

for any  $u, v, w \in V$ ,

$$(B4) \quad B^t(u(t), v(t)) \text{ is square integrable in } t \text{ on } [\tau, S] \text{ for } u(t), v(t) \in D(\partial\varphi^t), \quad (2.7)$$

where  $C_i$  ( $i = 4, 5, 6$ ) are positive constants and  $\theta_i \in [0, 1)$  ( $i = 4, 5, 6$ ).

**Remark 2.** Sometimes we write  $B^t(u, u) = B^t(u)$ .

Let  $R(t)$  be a linear operator mapping  $V$  into  $V'$  and  $D(\partial\varphi^t)$  into  $H$  such that

$$(R1) \quad \|R(t)u\| \leq C_7\|u\|_V^{1-\theta_7} \cdot \|g\|_H^{\theta_7} \quad \text{for any } u \in D(\partial\varphi^t), g \in \partial\varphi^t(u), \quad (2.8)$$

$$(R2) \quad |(R(t)u, u)| \leq C_8\|u\|_V^{1+\theta_8} \cdot \|u\|_H^{1-\theta_8} \quad \text{for any } u \in V, \quad (2.9)$$

$$(R3) \quad R(t)u(t) \text{ is square integrable in } t \text{ on } [\tau, S] \text{ for } u(t) \in D(\partial\varphi^t). \quad (2.10)$$

where  $C_i$  ( $i = 7, 8$ ) are positive constants and  $\theta_i \in [0, 1)$  ( $i = 7, 8$ ),

Then we introduce a nonlinear abstract evolution equation (AE) in  $H$  as follows:

$$(AE) \quad \frac{du}{dt} + \partial\varphi^t(u) + B^t(u, u) + R(t)u \ni f(t), \quad (2.11)$$

where  $f$  is given in  $H$  (concerning  $B^t$  and  $R(t)$ , see Temam<sup>14</sup>, Chapter III).

Now we state the definition of a strong solution of (AE).

**Definition 2.1** Let  $u : [\tau, S] \rightarrow H$ ,  $S \in (\tau, \infty)$ . Then  $u$  is called a strong solution of (AE) on  $[\tau, S]$  if it satisfies the following:

- (i)  $u \in C([\tau, S]; H)$ .
- (ii)  $u(t)$  is absolutely continuous on  $(\tau, S]$ .
- (iii)  $u(t) \in D(\partial\varphi^t)$  for a.e.  $t \in [\tau, S]$  and there exists  $g : [\tau, S] \rightarrow H$  satisfying  $g(t) \in \partial\varphi^t(u(t))$  and

$$\frac{du}{dt} + g(t) + B^t(u(t), u(t)) + R(t)u(t) = f(t) \quad \text{for a.e. } t \in [\tau, S]. \quad (2.12)$$

**Definition 2.2** Let  $u_\tau \in D(\varphi^\tau)$  or  $\overline{D(\varphi^\tau)}^H$ . If  $u(t)$  satisfies the equation (AE) and the condition  $u(\tau) = u_\tau$ , then it is called a strong solution of initial value problem for (AE).

Here we consider the following linear evolution equation (AE') in  $H$ . We will make use of solutions of (AE') to construct a strong solution of (AE).

$$(AE') \quad \frac{du}{dt} + \partial\varphi^t(u) \ni f(t), \quad (2.13)$$

where  $f$  is a given element in  $H$ .

A strong solution of (AE') is defined as follows:

**Definition 2.3** Let  $u : [\tau, S] \rightarrow H$ ,  $S \in (\tau, \infty)$ . Then  $u$  is called a strong solution of (AE') on  $[\tau, S]$  if it satisfies conditions (i), (ii) and (iii) of Definition 2.1 by replacing (2.12) by the following (2.14)

$$\frac{du}{dt} + g(t) = f(t) \quad \text{for a.e. } t \in [\tau, S]. \quad (2.14)$$

Now we prepare several terminologies which we use later.

**Definition 2.4** Let  $\tau \in \mathbb{R}$  be any fixed. Then

(i) (coercive). The operator  $\partial\varphi^t + R(t)$  is coercive if there exist a positive number  $\gamma$  and a nonnegative number  $C$  (both independent of  $\tau \in \mathbb{R}$ ) such that for any  $u \in D(\partial\varphi^t)$  and  $g \in \partial\varphi^t(u)$

$$(g, u)_H + (R(t)u, u)_H \geq \gamma \|u\|_V^2 - C \quad (2.15)$$

holds for every  $t \in [\tau, \infty)$ .

(ii) (semi-coersive). The operator  $\partial\varphi^t + R(t)$  is semi-coercive to (AE) if it satisfies the coercive type estimate to any strong solution of (AE), that is, there exist a positive number  $\gamma'$  and a nonnegative number  $C'$  (both independent of  $\tau \in \mathbb{R}$  and  $C'$  depends on  $\|u(\tau)\|_H$  but it is bounded provided  $\|u(\tau)\|_H$  goes around in a bounded set) such that for any strong solution  $u(t)$  of (AE) on  $[\tau, \infty)$  and  $g(t) \in \partial\varphi^t(u(t))$

$$(g(t), u(t))_H + (R(t)u(t), u(t))_H \geq \gamma' \|u(t)\|_V^2 - C'(\|u(\tau)\|_H) \quad (2.16)$$

holds for a.e.  $t \in [\tau, \infty)$ .

Now we state several theorems concerning with the uniqueness and existence of global strong solutions. Proofs of these theorems will be given in §3. Hereafter, concerning the equation (AE), we assume (A1)~(A5), (B0)~(B4) and (R1)~(R3).

**Theorem 2.1** *Let  $f \in L^\infty(-\infty, \infty; H)$  or  $f \in L^2(-\infty, \infty; H)$  and let  $\tau \in \mathbb{R}$ . For any  $u_\tau \in D(\varphi^\tau)$  or  $\overline{D(\varphi^\tau)}^H$ , there exists at most one strong solution of (AE) with  $u(\tau) = u_\tau$ .*

**Theorem 2.2** *Let  $f \in L^\infty(-\infty, \infty; H)$  or  $f \in L^2(-\infty, \infty; H)$  and let  $\tau \in \mathbb{R}$ . For any  $u_\tau \in D(\varphi^\tau)$ , there is a positive number  $\tau_0$  (depending on  $f$  and  $u_\tau$ ) such that a strong solution  $u$  of (AE) with  $u(\tau) = u_\tau$  uniquely exists on  $[\tau, \tau + \tau_0]$ . Moreover we find  $u(t)$  is absolutely continuous on  $[\tau, \tau + \tau_0]$  and  $du/dt \in L^2(\tau, \tau + \tau_0; H)$  and  $g \in L^2(\tau, \tau + \tau_0; H)$ , where  $g(t) \in \partial\varphi^t(u(t))$ . Furthermore  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau, \tau + \tau_0]$ .*

**Theorem 2.3** *Let  $f$  belong to  $L^\infty(-\infty, \infty; H)$  or  $L^2(\infty, \infty; H)$  and let  $\tau \in \mathbb{R}$  be any fixed. If the operator  $\partial\varphi^t + R(t)$  be coercive, then for any  $u_\tau \in \overline{D(\varphi^\tau)}^H$ , there exists a global strong solution  $u(t)$  of (AE) on  $[\tau, \infty)$  with  $u(\tau) = u_\tau$  and  $\varphi^t(u(t))$  is absolutely continuous on any finite interval  $[\delta, T] \subset (\tau, \infty)$ . We also find that  $u$  depends continuously on  $u_\tau$  in  $H$ . Moreover  $u$  is bounded with respect to  $V$ -norm on  $[\delta, \infty)$ . In particular, if  $u_\tau \in D(\varphi^\tau)$ , then  $u(t) \in D(\varphi^t)$  and  $\varphi^t(u(t))$  is absolutely continuous on any interval  $[\tau, T]$ , and moreover  $u$  is absolutely continuous on  $[\tau, T]$  and  $du/dt \in L^2(\tau, T; H)$ .*

**Theorem 2.4** *Suppose  $f$  belongs to  $L^\infty(-\infty, \infty; H)$  or  $L^2(-\infty, \infty; H)$  and let  $\tau \in \mathbb{R}$  be any fixed. If the operator  $\partial\varphi^t + R(t)$  be semi-coercive to (AE), then for any  $u_\tau \in \overline{D(\varphi^\tau)}^H$ , there exists a global strong solution  $u(t)$  of (AE) on  $[\tau, \infty)$  with  $u(\tau) = u_\tau$  and  $\varphi^t(u(t))$  is absolutely continuous on any finite interval  $[\delta, T] \subset (\tau, \infty)$ . We also see that  $u$  depends continuously on  $u_\tau$  in  $H$ . Moreover  $u$  is bounded with respect to  $V$ -norm on  $[\delta, \infty)$ . Particularly, if  $u_\tau \in D(\varphi^\tau)$ , then  $u(t) \in D(\varphi^t)$  and  $\varphi^t(u(t))$  is absolutely continuous on any interval  $[\tau, T]$ , and furthermore  $u$  is absolutely continuous on  $[\tau, T]$  and  $du/dt \in L^2(\tau, T; H)$ .*

### 3 Proofs of the existence theorems

#### 3.1 Some preliminary lemmas.

**Lemma 3.1** (the uniform Gronwall lemma) *Suppose  $p, q$  and  $y$  are positive, locally integrable functions on  $(t_0, \infty)$ . If the following differential inequality  $\frac{dy}{dt} \leq py + q$  holds for any  $t \geq t_0$  and if there are constants  $\alpha, a_i > 0$  ( $i = 1, 2, 3$ ) such that*

$$\int_t^{t+\alpha} p(s)ds \leq a_1, \quad \int_t^{t+\alpha} q(s)ds \leq a_2, \quad \int_t^{t+\alpha} y(s)ds \leq a_3$$

*for any  $t \geq t_0$ , then we have*

$$y(t + \alpha) \leq (a_3/\alpha + a_2)e^{a_1} \text{ for all } t \geq t_0.$$

PROOF OF LEMMA 3.1. See, for instance, Chapter III of the book<sup>14</sup>).

**Lemma 3.2** *Let  $B^t$  be the bilinear operator which satisfies (B0)~(B3) and let  $u, v \in D(\partial\varphi^t)$ . Put  $w = u - v$ , then we have*

$$\begin{aligned} & \|B^t(u, u) - B^t(v, v)\| \\ & \leq C_4(\|w\|_H^{\theta_4}\|w\|_V^{1-\theta_4}\|u\|_V^{1-\theta_4}\|\partial\varphi^t(u)\|_H^{\theta_4} + \|v\|_H^{\theta_4}\|v\|_V^{1-\theta_4}\|w\|_V^{1-\theta_4}\|\partial\varphi^t(w)\|_H^{\theta_4}). \end{aligned} \quad (3.1)$$

PROOF OF LEMMA 3.2. Using the bilinearity of  $B^t$ , we see

$$\|B^t(u, u) - B^t(v, v)\| = \|B^t(u - v, u) + B^t(v, u - v)\| \leq \|B^t(w, u)\| + \|B^t(v, w)\|$$

and by the assumption (B1), we get easily the desired inequality.

**Lemma 3.3** *Let  $B^t$  be the bilinear operator satisfying (B0)~(B3) and let  $u, v \in V$ . Putting  $w = u - v$ , then we have*

$$(B^t(u, u) - B^t(v, v), w) = (B^t(w, u), w), \quad (3.2)$$

$$|(B^t(u, u) - B^t(v, v), w)| \leq C_6\|w\|_H \cdot \|w\|_V \cdot \|u\|_V, \quad (3.3)$$

$$|(B^t(u, u) - B^t(v, v), w)| \leq C_4\|w\|_H^{1+\theta_4} \cdot \|w\|_V^{1-\theta_4} \cdot \|u\|_V^{1-\theta_4} \cdot \|\partial\varphi^t(u)\|_H^{\theta_4}. \quad (3.4)$$

PROOF OF LEMMA 3.3. First we show (3.2). Because of (B0), we can write

$$\begin{aligned} (B^t(u, u) - B^t(v, v), w) &= (B^t(u, u) - B^t(v, v), w) - (B^t(v, w), w) \\ &= (B^t(u, u), w) - (B^t(v, u), w) = (B^t(w, u), w). \end{aligned}$$

By means of (3.2), the inequality ((3.3)) (resp.(3.4)) follows from (B3) (resp.(B1)).

**Lemma 3.4** *Suppose the assumption (A4) holds. Given  $\tau < T$ . Let  $u : [\tau, T] \rightarrow H$  and  $\varphi^t(u(\cdot)) : [\tau, T] \rightarrow [0, \infty)$  be absolutely continuous on  $[\tau, T]$ . Let*

$$\mathcal{L} \equiv \{t \in (\tau, T); du/dt \text{ and } d\varphi^t(u(t))/dt \text{ both exist and } u(t) \in D(\partial\varphi^t)\}.$$

*Then there exist positive constants  $C_2$  and  $C_3$  such that*

$$|\frac{d}{dt}\varphi^t(u(t)) - (g, \frac{d}{dt}u(t))_H| \leq C_2\|g\|_H \cdot \varphi^t(u(t))^{\frac{1}{2}} + C_3\varphi^t(u(t)) \quad (3.5)$$

*for every  $t \in \mathcal{L}$  and  $g \in \partial\varphi^t(u(t))$ , where  $C_2$  and  $C_3$  are what appeared in (A4).*

PROOF OF LEMMA 3.4. Due to the assumption (A4), we can show the lemma (see Lemma 3.6 of Ôtani-Yamada<sup>12)</sup> and Proposition 3.2 of Yamada<sup>15)</sup>).

**Lemma 3.5** *Let  $u(t)$  be a strong solution of (AE) on  $[\tau, T]$ . Suppose there is a number  $\delta \in [\tau, T]$  such that  $u(t)$  is absolutely continuous on  $[\delta, T]$ ,  $du/dt \in L^2(\delta, T; H)$  and  $g \in L^2(\delta, T; H)$  where  $g \in \partial\varphi^t(u)$ . Then it holds that  $u(t) \in D(\varphi^t)$  for  $t \in [\delta, T]$  and  $\varphi^t(u(t))$  is absolutely continuous on  $[\delta, T]$ .*

PROOF OF LEMMA 3.5. To prove this lemma, we use the argument along by that of Yamada<sup>15)</sup>. For  $\lambda > 0$  and  $u \in H$ , we put

$$\varphi_\lambda^t(u) = \inf_{v \in H} \{ \varphi^t(v) + \frac{1}{2\lambda} \|u - v\|^2 \}.$$

Since  $u(t)$  is a strong solution,  $u(t) \in D(\partial\varphi^t)$  for a.e.  $t \in [\tau, T]$ . Using (1.4), (1.5) and (1.10) of the literature<sup>15)</sup>, then we find

$$\begin{aligned} \varphi_\lambda^t(u(t)) &\leq \varphi^t(u(t)) \quad \text{for a.e. } t \in [\tau, T], \\ \lim_{\lambda \searrow 0} \varphi_\lambda^t(u(t)) &= \varphi^t(u(t)) \quad \text{for a.e. } t \in [\tau, T], \\ \|\partial\varphi_\lambda^t(u(t))\| &\leq \|\partial\varphi^t(u(t))\| \quad \text{for a.e. } t \in [\tau, T]. \end{aligned}$$

Because of the absolute continuity of  $u(t)$  on  $[\delta, T]$  and the assumption (A4), we can apply Proposition 3.2 of the work<sup>15)</sup> to  $\varphi^t$ . Therefore we have for  $s, t \in [\delta, T]$

$$\begin{aligned} &|\varphi_\lambda^t(u(t)) - \varphi_\lambda^s(u(s))| \\ &\leq \int_s^t \{ \|\partial\varphi_\lambda^\sigma(u(\sigma))\| \cdot \left\| \frac{du(\sigma)}{d\sigma} \right\| + C_3 \varphi_\lambda^\sigma(u(\sigma)) + C_2 \|\partial\varphi_\lambda^\sigma(u(\sigma))\| \varphi_\lambda^\sigma(u(\sigma))^{\frac{1}{2}} \} d\tau \\ &\leq \int_s^t \{ \|\partial\varphi^\sigma(u(\sigma))\| \cdot \left\| \frac{du(\sigma)}{d\sigma} \right\| + C_3 \varphi^\sigma(u(\sigma)) + C_2 \|\partial\varphi^\sigma(u(\sigma))\| \varphi^\sigma(u(\sigma))^{\frac{1}{2}} \} d\tau \\ &\leq \int_s^t \{ \|\partial\varphi^\sigma(u(\sigma))\|^2 + \frac{1}{2} \left\| \frac{du(\sigma)}{d\sigma} \right\|^2 + (C_3 + \frac{C_2^2}{2}) \varphi^\sigma(u(\sigma)) \} d\sigma. \end{aligned} \tag{3.6}$$

If  $\lambda \rightarrow 0$  in (3.6), then we get for  $s, t \in [\delta, T]$

$$\begin{aligned} &|\varphi^t(u(t)) - \varphi^s(u(s))| \\ &\leq \int_s^t \{ \|\partial\varphi^\sigma(u(\sigma))\|^2 + \frac{1}{2} \left\| \frac{du(\sigma)}{d\sigma} \right\|^2 + (C_3 + \frac{C_2^2}{2}) \varphi^\sigma(u(\sigma)) \} d\sigma \\ &\equiv \int_s^t k(\sigma) d\sigma. \end{aligned} \tag{3.7}$$

By assumptions of the lemma, the integrand  $k(t)$  of the right hand side of (3.7) belongs to  $L^1(\delta, T)$ . In fact, it holds that

$$\begin{aligned} &\left| \int_\delta^T \varphi^\sigma(u(\sigma)) d\sigma \right| = \frac{1}{k} \left| \int_\delta^T (g(\sigma), u(\sigma))_H d\sigma \right| \\ &\leq \frac{1}{k} \int_\delta^T \|g(\sigma)\| \cdot \|u(\sigma)\| d\sigma \leq \frac{1}{2k} \int_\delta^T (\|g(\sigma)\|^2 + \|u(\sigma)\|^2) d\sigma < \infty, \end{aligned}$$

and other terms are also integrable on  $(\delta, T)$ . Thus we have shown that  $u(t) \in D(\varphi^t)$  for  $t \in [\delta, T]$  and  $\varphi^t(u(t))$  is absolutely continuous on  $[\delta, T]$ .

Before we refer to the existence of a local solution of (AE), we present the existence result of the strong solution of the linear equation (AE').

**Lemma 3.6** *Suppose assumptions (A1)~(A5) are fulfilled. Let  $f \in L^2(\tau, T; H)$  and  $u_\tau \in D(\varphi^\tau)$ . Then there exists a unique strong solution  $u$  of (AE)' on  $[\tau, T]$  with properties:*

- (i)  $u(t)$  is absolutely continuous on  $[\tau, T]$ .
- (ii)  $du/dt \in L^2(\tau, T; H)$  and  $g \in L^2(\tau, T; H)$  where  $g(t) \in \partial\varphi^t(u(t))$ .
- (iii)  $u(t) \in D(\varphi^t)$  for every  $t \in [\tau, T]$  and  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau, T]$ .
- (iv)  $u(\tau) = u_\tau$  in  $H$ .

PROOF OF LEMMA 3.6. We omit the proof. See the literature<sup>15)</sup>.

### 3.2 Uniqueness of of a strong solution

PROOF OF THEOREM 2.1.

For an initial data  $u_\tau \in \overline{D(\varphi^\tau)}$ , we assume that there exist strong solutions  $u_1$  and  $u_2$  of (AE) on  $[\tau, T]$  such that  $u_1(\tau) = u_2(\tau) = u_\tau$ . Then, by the definition of a strong solution of (AE), there are  $g_i(t) \in \partial\varphi^t(u_i(t))$  ( $i = 1, 2$ ) such that

$$\frac{du_i}{dt} + g_i(t) + B^t(u_i(t)) + R(t)u_i(t) = f(t) \quad (i = 1, 2) \quad (3.8)$$

for a.e.  $t \in [\tau, T]$ . Here we put  $w = u_1 - u_2$ , then by (A5) we have  $g_1(t) - g_2(t) \in \partial\varphi^t(w(t))$ . Moreover we see for a.e.  $t \in [\tau, T]$

$$\frac{dw}{dt} + g_1(t) - g_2(t) + B^t(u_1(t)) - B^t(u_2(t)) + R(t)w(t) = 0. \quad (3.9)$$

Now we prepare the following estimate

$$\begin{aligned} & |(B^t(u_1(t)) - B^t(u_2(t)), w(t))| \\ & \leq C_6 C_1^{-\frac{1}{2}} \|w(t)\|_H \varphi^t(w(t))^{\frac{1}{2}} \|u_1(t)\|_V \leq \frac{k}{4} \varphi^t(w(t)) + \frac{C_6^2}{k C_1} \|w(t)\|_H^2 \cdot \|u_1(t)\|_V^2, \end{aligned}$$

where we used (3.3) of Lemma 3.3; moreover by (R2) and Young's inequality with exponents  $1/r = (1 + \theta_8)/2$  and  $1/s = (1 - \theta_8)/2$  we find

$$|(R(t)w(t), w(t))| \leq C_8 (\varphi^t(w(t))/C_1)^{\frac{1+\theta_8}{2}} \|w(t)\|_H^{1-\theta_8} \leq \frac{k}{4} \varphi^t(w(t)) + \beta_8 \|w(t)\|_H^2,$$

where  $\beta_8 = \frac{1 - \theta_8}{2} \cdot \left(\frac{2(1 + \theta_8)}{k C_1}\right)^{\frac{1+\theta_8}{1-\theta_8}} C_8^{\frac{2}{1-\theta_8}}$ . Taking the scalar product of (3.9) with  $w(t)$  in  $H$  and making use of the above estimates, then we have

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_H^2 + \frac{k}{2} \varphi^t(w(t)) \leq \left(\frac{C_6^2}{k C_1} \|u_1(t)\|_V^2 + \beta_8\right) \|w(t)\|_H^2 \quad \text{for a.e. } t \in [\tau, T],$$

from which we obtain

$$\frac{d}{dt} \|w(t)\|_H^2 \leq 2 \left(\frac{C_6^2}{k C_1} \|u_1(t)\|_V^2 + \beta_8\right) \|w(t)\|_H^2 \quad \text{for a.e. } t \in [\tau, T].$$

Then, by the definition of a strong solution of (AE)' and Theorem I of the work<sup>15)</sup>,  $\varphi^t(u_1(t)) \in L^1(\tau, T)$ . Noting these facts and  $\|w(\tau)\|_H = \|u_1(\tau) - u_2(\tau)\|_H = 0$ , we have

$$\|w(t)\|_H^2 \leq \int_\tau^t 2 \left(\frac{C_6^2}{k C_1} \|u_1(s)\|_V^2 + \beta_8\right) \|w(s)\|_H^2 ds$$

for any  $t \in [\tau, T]$ . Thus, by usual Gronwall's inequality, we find  $\|w(t)\|_H = 0$  on  $[\tau, T]$ . Therefore we have obtained the uniqueness result.

### 3.3 Existence of a local solution

PROOF OF THEOREM 2.2.

Without loss of generality, we assume  $\|u_\tau\|_H + \|f\|_{L^2(\tau, T; H)} \neq 0$ . We define positive numbers  $M_i$  ( $i = 1, 2, 3$ ):

$$M_1 \equiv \varphi^\tau(u_\tau) + \frac{1}{2}(C_2^2 + C_3)\|u_\tau\|_H^2 + \frac{2}{k^2 C_1}(C_2^2 + C_3 + k^2 C_1)\|f\|_{L^2(\tau, T; H)}^2, \quad (3.10)$$

$$M_2 \equiv \frac{4C_4^2}{k^2 C_1^3}(C_2^2 + C_3 + k^2 C_1), \quad (3.11)$$

$$M_3 \equiv \frac{4C_7^2}{k^2 C_1^2}(C_2^2 + C_3 + k^2 C_1). \quad (3.12)$$

We claim that if we take a small positive number  $\tau_0$  satisfying

$$(10 + 12C_1)M_1 M_2 \tau_0^{1-\theta_4} + (10 + C_1)M_3 \tau_0^{1-\theta_7} < 1, \quad (3.13)$$

then there exists a strong solution of (AE) on  $[\tau, \tau + \tau_0]$  with  $u(\tau) = u_\tau$ .

To construct such a local solution, we employ an iteration (see works<sup>(12), (6)</sup>). We are going to make a pair of iterated sequences  $\{v_n\}$  and  $\{g_n\}$  by the following scheme:

$$\begin{cases} \frac{dv_1}{dt} + g_1(t) = f(t), & g_1(t) \in \partial\varphi^t(v_1(t)), \quad \text{a.e. } t \in [\tau, T], \\ v_1(\tau) = u_\tau, \end{cases} \quad (3.14)$$

$$\begin{cases} \frac{dv_n}{dt} + g_n(t) = f(t) - B^t(v_{n-1}(t)) - R(t)v_{n-1}(t), & \text{a.e. } t \in [\tau, T], \\ g_n(t) \in \partial\varphi^t(v_n(t)), & \text{a.e. } t \in [\tau, T], \\ v_n(\tau) = u_\tau, \end{cases} \quad (3.15)$$

where  $B^t(v_\ell(t)) = B^t(v_\ell(t), v_\ell(t))$ .

The proof consists of three steps:

I. (A priori estimate.) Putting

$$I_n = \sup_{\tau \leq t \leq \tau + \tau_0} \{ \varphi^t(v_n(t)) + \frac{1}{2} \int_\tau^t \|g_n(s)\|_H^2 ds \}, \quad (3.16)$$

then if  $\tau_0$  satisfies (3.13), the following estimate holds:

$$I_n \leq 2M_1 \quad \text{for all } n \geq 1. \quad (3.17)$$

II. (Convergence.)  $\{v_n\}$  and  $\{g_n\}$  form Cauchy sequences in  $C([\tau, \tau + \tau_0]; H) \cap L^\infty(\tau, \tau + \tau_0; V)$  and  $L^2(\tau, \tau + \tau_0; H)$  respectively.

III. (Existence.) The desired solution exists as the limit of  $\{v_n\}$ .

Step I. Thanks to Lemma 3.6, equations (3.14) and (3.15) have strong solutions for  $n \geq 1$ , and therefore we can carry out the iteration. Note that  $v_n(t) \in D(\partial\varphi^t)$  for a.e.  $t \in [\tau, T]$ . Now, to obtain an apriori estimate, we first take the scalar product of both sides of (3.15) with  $v_n$  in  $H$ , then we have

$$\frac{1}{2} \frac{d}{dt} \|v_n(t)\|^2 + k\varphi^t(v_n(t)) = (f(t), v_n(t)) - (B^t(v_{n-1}(t), v_n(t)) - (R(t)v_{n-1}(t), v_n(t)))$$



for a.e.  $t \in [\tau, T]$ , whence we have

$$\frac{1}{2} \frac{d}{dt} \|v_n(t)\|^2 + \frac{k}{2} \varphi^t(v_n(t)) \leq \frac{1}{2kC_1} \|E_{n-1}(t)\|^2 \quad \text{for a.e. } t \in [\tau, T], \quad (3.18)$$

where  $E_{n-1}(t) = f(t) - B^t(v_{n-1}(t)) - R(t)v_{n-1}(t)$  for  $n \geq 2$  and  $E_0(t) = f(t)$ . Integrating (3.18) on  $[\tau, t]$  with respect to  $t$ , then we obtain

$$\frac{1}{2} \|v_n(t)\|^2 + \frac{k}{2} \int_{\tau}^t \varphi^s(v_n(s)) ds \leq \frac{1}{2} \|u_{\tau}\|^2 + \frac{1}{2kC_1} \int_{\tau}^t \|E_{n-1}(s)\|^2 ds \quad (3.19)$$

for every  $t \in [\tau, T]$ . On the other hand, taking the scalar product of (3.15) with  $g_n$  in  $H$ , then we see

$$\begin{aligned} & \frac{d}{dt} \varphi^t(v_n(t)) + \|g_n(t)\|^2 \\ & \leq \|E_{n-1}(t)\| \cdot \|g_n(t)\| + C_2 \|g_n(t)\| \varphi^t(v_n(t))^{\frac{1}{2}} + C_3 \varphi^t(v_n(t)) \quad \text{for a.e. } t \in [\tau, T], \end{aligned} \quad (3.20)$$

where we used Lemma 3.4. Then (3.20) yields the following

$$\frac{d}{dt} \varphi^t(v_n(t)) + \frac{1}{2} \|g_n(t)\|^2 \leq (C_2^2 + C_3) \varphi^t(v_n(t)) + \|E_{n-1}(t)\|^2 \quad \text{for a.e. } t \in [\tau, T]. \quad (3.21)$$

Integrate (3.21) on  $[\tau, T]$ , then

$$\begin{aligned} & \varphi^t(v_n(t)) + \frac{1}{2} \int_{\tau}^t \|g_n(s)\|^2 ds \\ & \leq \varphi^{\tau}(v_n(\tau)) + (C_2^2 + C_3) \int_{\tau}^t \varphi^s(v_n(s)) ds + \int_{\tau}^t \|E_{n-1}(s)\|^2 ds \end{aligned} \quad (3.22)$$

holds for any  $t \in [\tau, T]$ . Here, multiplying (3.19) by  $2(C_2^2 + C_3)/k$  and adding the result to (3.22), then we find

$$\begin{aligned} I_n & \leq \varphi^{\tau}(u_{\tau}) + \frac{1}{k} (C_2^2 + C_3) \|u_{\tau}\|^2 \\ & \quad + \frac{1}{k^2 C_1} (C_2^2 + C_3 + k^2 C_1) \int_{\tau}^{\tau+\tau_0} \|E_{n-1}(s)\|^2 ds \quad \text{for all } n \geq 1. \end{aligned} \quad (3.23)$$

Here we note:

$$\begin{aligned} \|E_{n-1}(s)\|_H^2 & \leq 2\|f(s)\|_H^2 + 4\|B^s(v_{n-1}(s))\|_H^2 + 4\|R(s)v_{n-1}(s)\|_H^2, \\ \|B^s(v_{n-1}(s))\|_H & \leq C_4 \|v_{n-1}(s)\|_H^{\theta_4} \cdot \|v_{n-1}(s)\|_V^{2(1-\theta_4)} \cdot \|\partial \varphi^t(v_{n-1}(t))\|_H^{\theta_4}, \\ \|R(s)v_{n-1}(s)\|_H & \leq C_7 \|v_{n-1}(s)\|_V^{1-\theta_7} \cdot \|\partial \varphi^s(v_{n-1}(s))\|_H^{\theta_7}. \end{aligned}$$

Using these estimates, we have from (3.23)

$$\begin{aligned} I_n & \leq \varphi^{\tau}(u_{\tau}) + \frac{1}{k} (C_2^2 + C_3) \|u_{\tau}\|^2 + \frac{2}{k^2 C_1} (C_2^2 + C_3 + k^2 C_1) \|f\|_{L^2(\tau, T; H)}^2 \\ & \quad + \frac{4C_4^2}{k^2 C_1} (C_2^2 + C_3 + k^2 C_1) \int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_H^{2\theta_4} \cdot \|v_{n-1}(s)\|_V^{4(1-\theta_4)} \cdot \|\partial \varphi^s(v_{n-1}(s))\|_H^{2\theta_4} ds \\ & \quad + \frac{4C_7^2}{k^2 C_1} (C_2^2 + C_3 + k^2 C_1) \int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_V^{2(1-\theta_7)} \cdot \|\partial \varphi^s(v_{n-1}(s))\|_H^{2\theta_7} ds. \end{aligned} \quad (3.24)$$

In the right hand side of (3.24), the values of integrals are estimated as follows:

$$\int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_H^{2\theta_4} \cdot \|v_{n-1}(s)\|_V^{4(1-\theta_4)} \cdot \|\partial\varphi^s(v_{n-1}(s))\|_H^{2\theta_4} ds \leq \frac{1}{C_1^2} 2^{\theta_4} \tau_0^{1-\theta_4} I_{n-1}^2, \quad (3.25)$$

$$\int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_V^{2(1-\theta_7)} \cdot \|\partial\varphi^s(v_{n-1}(s))\|_H^{2\theta_7} ds \leq \frac{1}{C_1} 2^{\theta_7} \tau_0^{1-\theta_7} I_{n-1}. \quad (3.26)$$

Indeed, to obtain (3.25), we calculate as below:

$$\begin{aligned} & \int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_H^{2\theta_4} \cdot \|v_{n-1}(s)\|_V^{4(1-\theta_4)} \cdot \|\partial\varphi^s(v_{n-1}(s))\|_H^{2\theta_4} ds \\ & \leq \frac{1}{C_1^2} \int_{\tau}^{\tau+\tau_0} \varphi^s(v_{n-1}(s))^{2-\theta_4} \|g_{n-1}(s)\|_H^{2\theta_4} ds \\ & \leq \frac{1}{C_1^2} I_{n-1}^{2-\theta_4} \int_{\tau}^{\tau+\tau_0} \|g_{n-1}(s)\|_H^{2\theta_4} ds \\ & \leq \frac{1}{C_1^2} I_{n-1}^{2-\theta_4} \tau_0^{1-\theta_4} 2^{\theta_4} \left( \int_{\tau}^{\tau+\tau_0} \frac{1}{2} \|g_{n-1}(s)\|^2 ds \right)^{\theta_4} \leq \frac{1}{C_1^2} 2^{\theta_4} \tau_0^{1-\theta_4} I_{n-1}^2, \end{aligned}$$

where we used Hölder's inequality with  $1/p = (1 - \theta_4)$  and  $1/q = \theta_4$ .

While we estimate similarly

$$\begin{aligned} & \int_{\tau}^{\tau+\tau_0} \|v_{n-1}(s)\|_V^{2(1-\theta_7)} \cdot \|\partial\varphi^s(v_{n-1}(s))\|_H^{2\theta_7} ds \\ & \leq \frac{1}{C_1} \int_{\tau}^{\tau+\tau_0} \varphi^s(v_{n-1}(s))^{1-\theta_7} \|g_{n-1}(s)\|_H^{2\theta_7} ds \\ & \leq \frac{1}{C_1} I_{n-1}^{1-\theta_7} \int_{\tau}^{\tau+\tau_0} \|g_{n-1}(s)\|_H^{2\theta_7} ds \leq \frac{1}{C_1} 2^{\theta_7} \tau_0^{1-\theta_7} I_{n-1}, \end{aligned}$$

and thus we have (3.26).

By means of (3.25) and (3.26), we get for all  $n \geq 2$

$$I_n \leq M_1 + 2^{\theta_4} \tau_0^{1-\theta_4} M_2 I_{n-1}^2 + 2^{\theta_7} \tau_0^{1-\theta_7} M_3 I_{n-1}. \quad (3.27)$$

If  $\tau_0 > 0$  satisfies (3.13), by an elementary computation, we find

$$5 \cdot 2^{\theta_4} \tau_0^{1-\theta_4} M_1 M_2 + 25 \cdot 2^{2\theta_7} \tau_0^{2(1-\theta_7)} M_3^2 < 1. \quad (3.28)$$

Moreover the next inequality is clear:

$$2^{\theta_7} \tau_0^{1-\theta_7} M_3 I_{n-1} \leq 5 \cdot 2^{2\theta_7} \tau_0^{2(1-\theta_7)} M_3^2 I_{n-1}^2 / M_1 + \frac{1}{5} M_1. \quad (3.29)$$

Combining (3.27) and (3.29), and employing (3.28), then we have

$$\begin{aligned} I_n & \leq \frac{6}{5} M_1 + (2^{\theta_4} \tau_0^{1-\theta_4} M_2 + 5 \cdot 2^{2\theta_7} \tau_0^{2(1-\theta_7)} M_3^2 / M_1) I_{n-1}^2 \\ & < \frac{6}{5} M_1 + \frac{1}{5 M_1} I_{n-1}^2 \quad \text{for all } n \geq 2. \end{aligned} \quad (3.30)$$

On the other hand, it is clear that  $I_1 \leq M_1$ . Hence we can show from (3.30) that the desired a priori estimate (3.17) holds by induction with respect to  $n$ .

Step II. We will show that  $\{v_n\}$  and  $\{g_n\}$  are Cauchy sequences in  $C([\tau, \tau + \tau_0]; H) \cap L^\infty(\tau, \tau + \tau_0; V)$  and  $L^2(\tau, \tau + \tau_0; H)$  respectively. Put

$$\begin{aligned} w_n &= v_n - v_{n-1}, & h_n &= g_n - g_{n-1}, \\ B_n(t) &= B^t(v_n(t)) - B^t(v_{n-1}(t)), & R_n(t) &= R(t)(v_n(t) - v_{n-1}(t)), \end{aligned}$$

then thanks to (A5) we know that the following hold:

$$\begin{cases} h_n(t) \in \partial\varphi^t(w_n(t)) & \text{for a.e. } t \in [\tau, T], \\ \frac{d}{dt}w_n(t) + h_n(t) = -B_{n-1}(t) - R_{n-1}(t) & \text{for a.e. } t \in [\tau, T]. \end{cases} \quad (3.31)$$

Applying the same argument as what we used in getting (3.18) and (3.21), then we have

$$\frac{1}{2} \frac{d}{dt} \|w_n(t)\|_H^2 + \frac{k}{2} \varphi^t(w_n(t)) \leq \frac{1}{kC_1} (\|B_{n-1}(t)\|_H^2 + \|R_{n-1}(t)\|_H^2), \quad (3.32)$$

$$\frac{d}{dt} \varphi^t(w_n(t)) + \frac{1}{2} \|h_n(t)\|_H^2 \leq (C_2^2 + C_3) \varphi^t(w_n(t)) + 2\|B_{n-1}(t)\|_H^2 + 2\|R_{n-1}(t)\|_H^2 \quad (3.33)$$

for a.e.  $t \in [\tau, T]$ . Integrating (3.32) and (3.33) on  $[\tau, T]$ , multiplying the former by  $\frac{2}{k}(C_2^2 + C_3)$  and adding the result to the latter, then we obtain

$$\begin{aligned} & \varphi^t(w_n(t)) + \frac{1}{2} \int_\tau^t \|h_n(s)\|_H^2 ds \\ & \leq \frac{2}{k^2 C_1} (C_2^2 + C_3 + k^2 C_1) \int_\tau^{\tau+\tau_0} (\|B_{n-1}(s)\|_H^2 + \|R_{n-1}(s)\|_H^2) ds \end{aligned}$$

for each  $t \in [\tau, T]$ . Remembering Lemma 3.2 and the assumption on  $R(t)$ , then

$$\begin{aligned} & \varphi^t(w_n(t)) + \frac{1}{2} \int_\tau^t \|h_n(s)\|_H^2 ds \\ & \leq M_2 C_1^2 \int_\tau^{\tau+\tau_0} \{ \|w_{n-1}(s)\|_H^{2\theta_4} \|w_{n-1}(s)\|_V^{2(1-\theta_4)} \|v_{n-1}(s)\|_V^{2(1-\theta_4)} \|\partial\varphi^s(v_{n-1}(s))\|_H^{2\theta_4} \\ & \quad + \|v_{n-2}(s)\|_H^{2\theta_4} \|v_{n-2}(s)\|_V^{2(1-\theta_4)} \|w_{n-1}(s)\|_V^{2(1-\theta_4)} \|\partial\varphi^s(w_{n-1}(s))\|_H^{2\theta_4} \} ds \\ & \quad + \frac{M_3 C_1}{2} \int_\tau^{\tau+\tau_0} \|w_{n-1}(s)\|_V^{2(1-\theta_7)} \|\partial\varphi^s(w_{n-1}(s))\|_H^{2\theta_7} ds \end{aligned} \quad (3.34)$$

for any  $t \in [\tau, T]$ , where  $M_2 = \frac{4C_4^2}{k^2 C_1^3} (C_2^2 + C_3 + k^2 C_1)$  and  $M_3 = \frac{4C_7^2}{k^2 C_1^2} (C_2^2 + C_3 + k^2 C_1)$  (see (3.11) and (3.12)). Here we put

$$\|u\|_{2,S} \equiv \left( \int_\tau^S \|u(t)\|_H^2 dt \right)^{1/2}, \quad \|u\|_{\infty,S} \equiv \sup_{\tau \leq t \leq S} \|u(t)\|_V,$$

and recall the definition of  $I_n$  and the a priori estimate (3.17), then it follows from (3.34) that

$$\varphi^t(w_n(t)) + \frac{1}{2} \int_\tau^t \|h_n(s)\|_H^2 ds$$

$$\begin{aligned}
 &\leq M_2 C_1^2 \{ \|w_{n-1}\|_{\infty, \tau_0}^2 \cdot C_1^{-(1-\theta_4)} I_{n-1}^{1-\theta_4} \left( \int_{\tau}^{\tau+\tau_0} \frac{1}{2} \|g_{n-1}(s)\|_H^2 ds \right)^{\theta_4} \left( \int_{\tau}^{\tau+\tau_0} 2^{\frac{\theta_4}{1-\theta_4}} ds \right)^{1-\theta_4} \\
 &\quad + C_1^{-1} I_{n-2} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_4)} \left( \int_{\tau}^{\tau+\tau_0} \|h_{n-1}(s)\|_H^2 ds \right)^{\theta_4} \left( \int_{\tau}^{\tau+\tau_0} 1^{\frac{1}{1-\theta_4}} ds \right)^{1-\theta_4} \} \\
 &\quad + \frac{M_3 C_1}{2} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_7)} \left( \int_{\tau}^{\tau+\tau_0} \|h_{n-1}(s)\|_H^2 ds \right)^{\theta_7} \left( \int_{\tau}^{\tau+\tau_0} 1^{\frac{1}{1-\theta_7}} ds \right)^{1-\theta_7} \\
 &\leq 2M_1 M_2 C_1^2 \tau_0^{1-\theta_4} (2^{\theta_4} C_1^{-(1-\theta_4)} \|w_{n-1}\|_{\infty, \tau_0}^2 + C_1^{-1} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_4)} \|h_{n-1}\|_{2, \tau_0}^{2\theta_4}) \\
 &\quad + \frac{M_3 C_1}{2} \tau_0^{1-\theta_7} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_7)} \|h_{n-1}\|_{2, \tau_0}^{2\theta_7} \quad \text{for any } t \in [\tau, \tau + \tau_0]. \tag{3.35}
 \end{aligned}$$

Since  $C_1 \|u\|_V^2 \leq \varphi^t(u)$ , we see from (3.35) that

$$\begin{aligned}
 &\|w_n\|_{\infty, \tau_0}^2 + \|h_n\|_{2, \tau_0}^2 \\
 &\leq (2M_1 M_2 C_1 + 4M_1 M_2 C_1^2) \tau_0^{1-\theta_4} (2^{\theta_4} C_1^{-(1-\theta_4)} \|w_{n-1}\|_{\infty, \tau_0}^2 + C_1^{-1} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_4)} \|h_{n-1}\|_{2, \tau_0}^{2\theta_4}) \\
 &\quad + \left( \frac{M_3}{2} + M_3 C_1 \right) \tau_0^{1-\theta_7} \|w_{n-1}\|_{\infty, \tau_0}^{2(1-\theta_7)} \|h_{n-1}\|_{2, \tau_0}^{2\theta_7} \\
 &\leq (2M_1 M_2 C_1 + 4M_1 M_2 C_1^2) \tau_0^{1-\theta_4} (2^{\theta_4} C_1^{-(1-\theta_4)} \|w_{n-1}\|_{\infty, \tau_0}^2 + C_1^{-1} (1 - \theta_4) \|w_{n-1}\|_{\infty, \tau_0}^2 \\
 &\quad + C_1^{-1} \theta_4 \|h_{n-1}\|_{2, \tau_0}^2) + \left( \frac{M_3}{2} + M_3 C_1 \right) \tau_0^{1-\theta_7} ((1 - \theta_7) \|w_{n-1}\|_{\infty, \tau_0}^2 + \theta_7 \|h_{n-1}\|_{2, \tau_0}^2),
 \end{aligned}$$

where we used an elementary inequality  $a^{2(1-\theta_i)} b^{2\theta_i} \leq (1 - \theta_i) a^2 + \theta_i b^2$ .

We notice that we can assume  $C_1 \leq 1$  without loss of generality (see (A2)), so we see  $2^{\theta_4} C_1^{-(1-\theta_4)} \leq 2/C_1$ . Therefore

$$\begin{aligned}
 &\|w_n\|_{\infty, \tau_0}^2 + \|h_n\|_{2, \tau_0}^2 \\
 &\leq (2M_1 M_2 + 4M_1 M_2 C_1) \tau_0^{1-\theta_4} (3 \|w_{n-1}\|_{\infty, \tau_0}^2 + \|h_{n-1}\|_{2, \tau_0}^2) \\
 &\quad + \left( \frac{M_3}{2} + M_3 C_1 \right) \tau_0^{1-\theta_7} (\|w_{n-1}\|_{\infty, \tau_0}^2 + \|h_{n-1}\|_{2, \tau_0}^2) \\
 &\leq \{6(1 + 2C_1) M_1 M_2 \tau_0^{1-\theta_4} + \left( \frac{1}{2} + C_1 \right) M_3 \tau_0^{1-\theta_7}\} (\|w_{n-1}\|_{\infty, \tau_0}^2 + \|h_{n-1}\|_{2, \tau_0}^2).
 \end{aligned}$$

Recall that we took  $\tau_0$  satisfying (3.13). Consequently, we find

$$6(1 + 2C_1) M_1 M_2 \tau_0^{1-\theta_4} + \left( \frac{1}{2} + C_1 \right) M_3 \tau_0^{1-\theta_7} < 1,$$

whence we can show that  $\{v_n\}$  (resp.  $\{g_n\}$ ) is a Cauchy sequence in  $C([\tau, \tau + \tau_0]; H) \cap L^\infty(\tau, \tau + \tau_0; V)$  (resp. in  $L^2(\tau, \tau + \tau_0; H)$ ).

Step III. Finally we will show the existence of a local solution of (AE). In view of the result of Step II, together with Lemma 3.2 and (R1), we see that sequences  $\{B^t(v_n)\}$  and  $\{R(t)v_n\}$  are both Cauchy sequences in  $L^2(\tau, \tau + \tau_0; H)$ . Therefore it follows from (3.15) that  $\{\frac{dv_n}{dt}\}$  also forms a Cauchy sequence in  $L^2(\tau, \tau + \tau_0; H)$ . Let us put:

$$\lim_{n \rightarrow \infty} v_n = v \quad \text{in } C([\tau, \tau + \tau_0]; H) \cap L^\infty(\tau, \tau + \tau_0; V), \tag{3.36}$$

$$\lim_{n \rightarrow \infty} g_n = g \quad \text{in } L^2(\tau, \tau + \tau_0; H). \tag{3.37}$$

Then we will show that:

$$v(t) \in D(\partial\varphi^t) \quad \text{and } g \in \partial\varphi^t(v(t)) \text{ for a.e. } t \in [\tau, \tau + \tau_0], \quad (3.38)$$

$$B^t(v_n) \rightarrow B^t(v) \quad \text{in } L^2(\tau, \tau + \tau_0; H) \text{ as } n \rightarrow \infty, \quad (3.39)$$

$$R(t)(v_n) \rightarrow R(t)(v) \quad \text{in } L^2(\tau, \tau + \tau_0; H) \text{ as } n \rightarrow \infty. \quad (3.40)$$

First we show (3.38). We note that  $v_n(t) \rightarrow v(t)$  in  $V$  for all  $t \in [\tau, \tau + \tau_0]$  and there exists a subsequence  $\{g_{n'}\}$  such that  $g_{n'}(t) \rightarrow g(t)$  in  $H$  for a.e.  $t \in [\tau, \tau + \tau_0]$ . Therefore, by means of a well-known property of a maximal monotone operator, we ensure that (3.38) is valid.

Next we will verify (3.39). In fact, according to Lemma 3.2, we have for  $t \in [\tau, \tau + \tau_0]$

$$\begin{aligned} & \left\| \int_{\tau}^t B^s(v_{n-1}(s))ds - \int_{\tau}^t B^s(v(s))ds \right\|_H \\ & \leq C_4 \int_{\tau}^t (\|v_{n-1}(s) - v(s)\|_H^{\theta_4} \|v_{n-1}(s) - v(s)\|_V^{1-\theta_4} \|v_{n-1}(s)\|_V^{1-\theta_4} \|g_{n-1}(s)\|_H^{\theta_4} \\ & \quad + \|v(s)\|_H^{\theta_4} \|v(s)\|_V^{1-\theta_4} \|v_{n-1}(s) - v(s)\|_V^{1-\theta_4} \|g_{n-1}(s) - g(s)\|_H^{\theta_4}) ds. \end{aligned}$$

By (3.36), we see that  $\|v_{n-1}(s) - v(s)\|_V$  and  $\|v_{n-1}(s) - v(s)\|_H$  tend to 0 for every  $s \in [\tau, \tau + \tau_0]$  as  $n \rightarrow \infty$ . Moreover  $\|v_{n-1}(s)\|_V$  is bounded in  $n$  and  $s$ . While we find that by elementary calculation

$$\begin{aligned} \int_{\tau}^t \|g_{n-1}(s)\|_H^{\theta_4} ds & \leq \left( \int_{\tau}^t \|g_{n-1}(s)\|_H^2 ds \right)^{\frac{\theta_4}{2}} \tau_0^{\frac{2-\theta_4}{2}} \text{ for any } t \in [\tau, \tau + \tau_0], \\ \int_{\tau}^t \|g_{n-1}(s) - g(s)\|_H^{\theta_4} ds & \leq \left( \int_{\tau}^t \|g_{n-1}(s) - g(s)\|_H^2 ds \right)^{\frac{\theta_4}{2}} \tau_0^{\frac{2-\theta_4}{2}} \text{ for any } t \in [\tau, \tau + \tau_0]. \end{aligned}$$

Thanks to these estimates, we can show easily that (3.39) holds.

As for (3.40), the assumption (R1) lead us to the following:

$$\left\| \int_{\tau}^t (R(s)v_{n-1}(s) - R(s)v(s))ds \right\|_H \leq C_7 \int_{\tau}^t \|v_{n-1}(s) - v(s)\|_V^{1-\theta_7} \|g_{n-1}(s) - g(s)\|_H^{\theta_7} ds. \quad (3.41)$$

It is an immediately consequence from (3.41) that (3.40) holds.

Now, integrating (3.15) on  $[\tau, t]$ , then we have

$$v_n(t) + \int_{\tau}^t g_n(s)ds = u_{\tau} + \int_{\tau}^t f(s)ds - \int_{\tau}^t B^s(v_{n-1}(s))ds - \int_{\tau}^t R(s)v_{n-1}(s)ds$$

for any  $t \in [\tau, \tau + \tau_0]$ . By virtue of (3.36), (3.37), (3.39) and (3.40), we can obtain

$$v(t) + \int_{\tau}^t g(s)ds = u_{\tau} + \int_{\tau}^t f(s)ds - \int_{\tau}^t B^s(v(s))ds - \int_{\tau}^t R(s)v(s)ds$$

for every  $t \in [\tau, \tau + \tau_0]$ . Hence  $v(t)$  is absolutely continuous on  $[\tau, \tau + \tau_0]$  and we see that  $\frac{dv_n}{dt}$  converges to  $\frac{dv}{dt}$  in  $L^2(\tau, \tau + \tau_0; H)$ . Thus we have shown that  $v(t)$  is a local solution of (AE) on  $[\tau, \tau + \tau_0]$ . As for that  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau, \tau + \tau_0]$ , we can show easily by means of Lemma 3.5. Though the proof of the uniqueness part remains, but it is essentially the same as that of Theorem 2.1, so we omit it.

### 3.4 Some lemmas (continued)

We state some lemmas to obtain global solutions. We start with two a priori estimates.

**Lemma 3.7** *Let the operator  $\partial\varphi^t + R(t)$  be coercive and let  $u(t)$  be any strong solution of (AE) on  $[\tau, T]$  where  $\tau < T$  be arbitrary. Then we have the following a priori estimates.*

(i) *If  $f \in L^2(-\infty, \infty; H)$ , then*

$$\|u(t)\|_H^2 \leq e^{-\gamma(t-\tau)} \|u(\tau)\|_H^2 + \frac{1}{\gamma} \|f\|_{L^2(-\infty, \infty; H)}^2 + \frac{2C}{\gamma} (1 - e^{-\gamma(t-\tau)}) \quad \text{for every } t \in [\tau, T]. \quad (3.42)$$

(ii) *If  $f \in L^\infty(-\infty, \infty; H)$ , then*

$$\|u(t)\|_H^2 \leq e^{-\gamma(t-\tau)} \|u(\tau)\|_H^2 + \left(\frac{1}{\gamma^2} \|f\|_{L^\infty(-\infty, \infty; H)}^2 + \frac{2C}{\gamma}\right) (1 - e^{-\gamma(t-\tau)}) \quad \text{for every } t \in [\tau, T]. \quad (3.43)$$

Here  $\gamma$  and  $C$  are what appeared in (i) of Definition 2.4.

PROOF OF LEMMA 3.7. Since  $u(t)$  is a strong solution of (AE), it satisfies (2.12). Multiply (2.12) by  $u(t)$  and use the assumption (B0), then we have

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + (g(t), u(t)) + (R(t)u(t), u(t)) = (f(t), u(t))$$

for a.e.  $t \in [\tau, T]$  and we find by the coercivity

$$\frac{d}{dt} \|u(t)\|_H^2 + \gamma \|u(t)\|_H^2 \leq \frac{1}{\gamma} \|f(t)\|_H^2 + 2C \quad \text{for every } t \in [\tau, T], \quad (3.44)$$

where we used  $\|u\|_H \leq \|u\|_V$ . If  $f \in L^2(-\infty, \infty; H)$ , integrating on  $[\tau, t]$  and considering  $e^{-\gamma(t-s)} \leq 1$ , we get (3.42) and if  $f \in L^\infty(-\infty, \infty; H)$ , since  $\|f(s)\|_H$  is majorized by  $\|f\|_{L^\infty(-\infty, \infty; H)}^2$ , the estimate (3.43) can be obtained immediately.

**Lemma 3.8** *Suppose the operator  $\partial\varphi^t + R(t)$  is semi-coercive to (AE). Let  $u(t)$  be any strong solution of (AE) on  $[\tau, T]$  where  $\tau < T$  be arbitrary. Let  $\gamma'$  and  $C' = C'(\|u(\tau)\|_H)$  stand for constants in (ii) of Definition 2.4. Then the following a priori estimates hold.*

(i) *If  $f \in L^2(-\infty, \infty; H)$ , then we have for every  $t \in [\tau, T]$*

$$\|u(t)\|_H^2 \leq e^{-\gamma'(t-\tau)} \|u(\tau)\|_H^2 + \frac{1}{\gamma'} \|f\|_{L^2(-\infty, \infty; H)}^2 + \frac{2C'}{\gamma'} (1 - e^{-\gamma'(t-\tau)}). \quad (3.45)$$

(ii) *If  $f \in L^\infty(-\infty, \infty; H)$ , then we get for every  $t \in [\tau, T]$*

$$\|u(t)\|_H^2 \leq e^{-\gamma'(t-\tau)} \|u(\tau)\|_H^2 + \left(\frac{1}{(\gamma')^2} \|f\|_{L^\infty(-\infty, \infty; H)}^2 + \frac{2C'}{\gamma'}\right) (1 - e^{-\gamma'(t-\tau)}). \quad (3.46)$$

PROOF OF LEMMA 3.8. By the same computation to that of Lemma 3.7, we have

$$\frac{d}{dt} \|u(t)\|_H^2 + \gamma' \|u(t)\|_H^2 \leq \frac{1}{\gamma'} \|f(t)\|_H^2 + 2C' \quad \text{for any } t \in [\tau, T]. \quad (3.47)$$

From this inequality, we have desired estimates at once.

The following lemma is important for us to show the existence of a global strong solution.

**Lemma 3.9** Assume  $f$  belongs to  $L^\infty(-\infty, \infty; H)$  or  $L^2(-\infty, \infty; H)$ . Let  $u(t)$  be a strong solution of (AE) on  $[\tau, T]$  where  $\tau < T$  be arbitrary. Moreover we suppose:

(i) For  $u(t)$ , there exist  $A_0 > 0$  and  $A'_0 > 0$ , independent of  $T$ , such that the following a priori estimate holds:

$$\|u(t)\|_H^2 \leq A_0 + A'_0 \|u(\tau)\|_H^2 \quad \text{for any } t \in [\tau, T]. \quad (3.48)$$

(ii)  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau + \eta, T]$  for some  $\eta \in [0, T - \tau]$ . Then, for any  $\delta \in (0, T)$  satisfying  $\tau + \eta + \delta < T$ , there exist  $a_i(\delta) > 0$  ( $i = 1, 2, 3$ ), independent of  $\eta$  and  $T$ , such that

$$\varphi^t(u(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta)) \quad \text{for all } t \in [\tau + \eta + \delta, T]. \quad (3.49)$$

PROOF OF LEMMA 3.9. Taking the scalar product of (AE) with  $g$  in  $H$ , and using Lemma 3.4, (B1) and (R1), then we have

$$\begin{aligned} & \frac{d}{dt} \varphi^t(u(t)) + \|g(t)\|_H^2 \\ & \leq -(B^t(u(t)), g(t))_H - (R(t)u(t), g(t))_H + (f(t), g(t))_H \\ & \quad + C_2 \|g(t)\|_H \cdot \varphi^t(u(t))^{1/2} + C_3 \varphi^t(u(t)) \\ & \leq C_4 \|u(t)\|_H^{\theta_4} \|u(t)\|_V^{1-\theta_4} \|u(t)\|_V^{1-\theta_4} \|g(t)\|_H^{1+\theta_4} + C_7 \|u(t)\|_V^{1-\theta_7} \|g(t)\|_H^{1+\theta_7} \\ & \quad + \|f(t)\|_H \cdot \|g(t)\|_H + C_2 \|g(t)\|_H \varphi^t(u(t))^{1/2} + C_3 \varphi^t(u(t)) \\ & \leq 4 \times \frac{1}{8} \|g(t)\|_H^2 + \frac{1-\theta_4}{2} \{4(1+\theta_4)\}^{\frac{1+\theta_4}{1-\theta_4}} \cdot C_4^{\frac{2}{1-\theta_4}} \|u(t)\|_H^{\frac{2\theta_4}{1-\theta_4}} \|u(t)\|_V^4 \\ & \quad + \frac{1-\theta_7}{2} \{4(1+\theta_7)\}^{\frac{1+\theta_7}{1-\theta_7}} \cdot C_7^{\frac{2}{1-\theta_7}} \|u(t)\|_V^2 + 2\|f(t)\|_H^2 + (2C_2^2 + C_3) \varphi^t(u(t)) \end{aligned} \quad (3.50)$$

for a.e.  $t \in [\tau + \eta, T]$ , where we used Young's inequality with exponents  $1/p = (1 + \theta_i)/2$  and  $1/q = (1 - \theta_i)/2$ . We write  $\alpha_i = \frac{1-\theta_i}{2} \{4(1+\theta_i)\}^{\frac{1+\theta_i}{1-\theta_i}} \cdot C_i^{\frac{2}{1-\theta_i}}$  ( $i = 4, 7$ ). Recalling the assumption (3.48), we see from (3.50) that

$$\begin{aligned} & \frac{d}{dt} \varphi^t(u(t)) + \frac{1}{2} \|g(t)\|_H^2 \\ & \leq \alpha_4 \|u(t)\|_H^{\frac{2\theta_4}{1-\theta_4}} \|u(t)\|_V^4 + \alpha_7 \|u(t)\|_V^2 + (2C_2^2 + C_3) \varphi^t(u(t)) + 2\|f(t)\|_H^2 \\ & \leq \{\alpha_4 (A_0 + A'_0 \|u(\tau)\|_H^2)^{\frac{\theta_4}{1-\theta_4}} C_1^{-2} \varphi^t(u(t)) + \frac{\alpha_7}{C_1} + 2C_2^2 + C_3\} \varphi^t(u(t)) + 2\|f(t)\|_H^2 \end{aligned} \quad (3.51)$$

holds for a.e.  $t \in [\tau + \eta, T]$ .

Here we are going to use Gronwall's inequality. In Lemma 3.1, we set  $\alpha = \delta$ ,  $y = \varphi^t(u(t))$ ,  $q = 2\|f(t)\|_H^2$  and

$$p = \alpha_4 (A_0 + A'_0 \|u(\tau)\|_H^2)^{\frac{\theta_4}{1-\theta_4}} C_1^{-2} \varphi^t(u(t)) + \frac{\alpha_7}{C_1} + 2C_2^2 + C_3.$$

Easily we see

$$\int_t^{t+\delta} q(s) ds = \int_t^{t+\delta} 2\|f(s)\|_H^2 ds \leq m(f) \equiv a_2(\delta),$$

where  $m(f) = 2\delta\|f\|_{L^\infty(-\infty,\infty;H)}^2$  if  $f \in L^\infty(-\infty,\infty;H)$ , and  $m(f) = 2\|f\|_{L^2(-\infty,\infty;H)}^2$  if  $f \in L^2(-\infty,\infty;H)$ .

Let us estimate  $y(t) = \varphi^t(u(t))$ . To do this, taking the scalar product of (AE) with  $u(t)$  in  $H$  and considering (B0), (R2), then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + k\varphi^t(u(t)) \\ & \leq C_8 \|u(t)\|_V^{1+\theta_8} \|u(t)\|_H^{1-\theta_8} + \|f(t)\|_H \cdot \|u(t)\|_H \\ & \leq \frac{kC_1}{2} \|u(t)\|_V^2 + \frac{1-\theta_8}{2} \left( \frac{1+\theta_8}{kC_1} \right)^{\frac{1+\theta_8}{1-\theta_8}} C_8^{\frac{2}{1-\theta_8}} \|u(t)\|_H^2 + \|u(t)\|_H^2 + \frac{1}{4} \|f(t)\|_H^2 \end{aligned} \quad (3.52)$$

for a.e.  $t \in [\tau + \eta, T]$ . In the above calculation, we used Young's inequality with exponents  $1/r = (1 + \theta_8)/2$  and  $1/s = (1 - \theta_8)/2$ . Put  $\alpha_8 = \frac{1 - \theta_8}{2} \left( \frac{1 + \theta_8}{kC_1} \right)^{\frac{1+\theta_8}{1-\theta_8}} C_8^{\frac{2}{1-\theta_8}}$ . Since  $\varphi^t(u(t)) \geq C_1 \|u(t)\|_V^2$ , with the aid of (3.48), we obtain from (3.52)

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + \frac{k}{2} \varphi^t(u(t)) \leq (\alpha_8 + 1)(A_0 + A'_0 \|u(\tau)\|_H^2) + \frac{1}{4} \|f(t)\|_H^2 \quad (3.53)$$

for a.e.  $t \in [\tau + \eta, T]$ . Integrating both sides of (3.53) on  $[t, t + \delta]$  with respect to  $t$  and using (3.48) again, then we have

$$\|u(t + \delta)\|_H^2 + k \int_t^{t+\delta} \varphi^s(u(s)) ds \leq (2(\alpha_8 + 1)\delta + 1)(A_0 + A'_0 \|u(\tau)\|_H^2) + n(f)$$

for every  $t \in [\tau + \eta, T - \delta]$ , where  $n(f)$  stands for  $\frac{\delta}{2} \|f\|_{L^\infty(-\infty,\infty;H)}^2$  if  $f \in L^\infty(-\infty,\infty;H)$ , and  $\frac{1}{2} \|f\|_{L^2(-\infty,\infty;H)}^2$  if  $f \in L^2(-\infty,\infty;H)$ .

Hence we have for any  $t \in [\tau + \eta, T - \delta]$

$$\int_t^{t+\delta} y(s) ds = \int_t^{t+\delta} \varphi^s(u(s)) ds \leq \frac{1}{k} \{ (2(\alpha_8 + 1)\delta + 1)(A_0 + A'_0 \|u(\tau)\|_H^2) + n(f) \} \equiv a_3(\delta).$$

and furthermore we get

$$\int_t^{t+\delta} p(s) ds \leq \frac{\alpha_4}{C_1^2} \cdot (A_0 + A'_0 \|u(\tau)\|_H^2)^{\frac{\theta_4}{1-\theta_4}} \cdot a_3(\delta) + \frac{\alpha_7}{C_1} + 2C_2^2 + C_3 \equiv a_1(\delta)$$

for any  $t \in [\tau + \eta, T - \delta]$ . Therefore we can obtain (3.49) by Gronwall's inequality and thus we proved the lemma.

### 3.5 Existence of global solutions

PROOF OF THEOREM 2.3.

First we assume  $u_\tau \in D(\varphi^\tau)$ . Then, thanks to Theorem 2.2, there exists a local solution  $u$  of (AE) on some interval  $[\tau, \tau + \tau_0]$ . Since the operator  $\partial\varphi^t + R(t)$  is coercive, we have an a priori estimate by Lemma 3.7, that is,  $\|u(t)\|_H^2 \leq A_0 + A'_0 \|u(\tau)\|_H^2$  for all  $t \geq \tau$ , where  $A_0 = \frac{1}{\gamma} \|f\|_{L^2(-\infty,\infty;H)} + \frac{2C}{\gamma}$ ,  $A'_0 = 1$  if  $f \in L^2(-\infty,\infty;H)$  and  $A_0 = \frac{1}{\gamma^2} \|f\|_{L^\infty(-\infty,\infty;H)} + \frac{2C}{\gamma}$ ,



$A'_0 = 1$  if  $f \in L^\infty(-\infty, \infty; H)$ . Moreover we find  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau, \tau + \tau_0]$  by Theorem 2.2. So, Lemma 3.9 is applicable to our local solution  $u$ . Consequently, we have for  $\delta > 0$

$$\varphi^t(u(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta)) \quad \text{for all } t \in [\tau + \delta, \tau + \tau_0]. \quad (3.54)$$

Using Theorem 2.2 again, we have an extension of  $u$  to  $[\tau, \tau + \tau_0 + \tau_1]$  and we find

$$\varphi^t(u(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta)) \quad \text{for all } t \in [\tau + \delta, \tau + \tau_0 + \tau_1],$$

where  $\tau_1 > 0$  is determined by the magnitude  $\varphi^{\tau+\tau_0}(u(\tau + \tau_0))$  which is bounded by the right hand side of (3.54). Since  $a_i(\delta)$  ( $i = 1, 2, 3$ ) are independent of  $\tau_0$  and  $\tau_1$ , therefore repeating such a procedure, we can extend  $u$  on  $[\tau, \infty)$ . Furthermore, owing to Theorem 2.2,  $u(t)$  has desired properties.

Next let  $u_\tau \in \overline{D(\varphi^\tau)}^H$ . Then we can take a sequence  $\{u_{\tau n}\} \subset D(\varphi^\tau)$  such that  $\|u_{\tau n} - u_\tau\|_H \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $T > 0$  be arbitrary and fixed for a time. According to the previous argument in this proof, we see that for each  $u_{\tau n}$ , there exists a strong solution  $u_n$  of (AE) on  $[\tau, T]$  with  $u_n(\tau) = u_{\tau n}$ . Note that there exists a constant  $M = M(u_\tau) > 0$  such that  $\|u_{\tau m}\|_H \leq M$  for all  $m \geq 1$  and remember that by virtue of Lemma 3.7 the a priori estimate  $\|u_m(t)\|_H^2 \leq A_0 + A'_0 \|u_{\tau m}\|_H^2$  ( $m \geq 1$ ) holds. Then by Gronwall inequality we get on  $[\tau, T]$ :

$$\begin{aligned} \|u_n(t) - u_m(t)\|_H^2 &\leq \|u_{\tau n} - u_{\tau m}\|_H^2 \exp \left\{ \int_\tau^t 2 \left( \frac{C_6^2}{kC_1} \|u_m(s)\|_V^2 + \beta_8 \right) ds \right\} \leq \|u_{\tau n} - u_{\tau m}\|_H^2 \exp C(\tau, T) \\ &\rightarrow 0 \text{ as } n, m \rightarrow \infty, \quad C(\tau, T) = \frac{2C_6^2}{k^2C_1^2} \left[ \left\{ 2(\alpha_8 + 1)(T - \tau) + 1 \right\} (A_0 + A'_0 M^2) + n(f) \right] + 2\beta_8(T - \tau), \end{aligned}$$

where  $\beta_8$  is what appeared in the proof of Theorem 2.1. Therefore there exists  $u \in C([\tau, T]; H)$  such that  $\|u_n(t) - u(t)\|_H \rightarrow 0$  uniformly on  $[\tau, T]$  as  $n \rightarrow \infty$ . Now we will show  $u$  is a strong solution of (AE). To do this, using the same argument as the proof of Lemma 3.5, we see that  $\varphi^t(u_n(t))$  is absolutely continuous on  $[\tau, T]$ . Here we use again the a priori estimate  $\|u_n(t)\|_H^2 \leq A_0 + A'_0 \|u_{\tau n}\|_H^2$  ( $t \in [\tau, T]$ ). Then these imply that for  $n \geq 1$

$$\varphi^t(u_n(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta)) \quad \text{for any } t \in [\tau + \delta, T].$$

In the above inequality,  $a_i(\delta)$  ( $i = 1, 2, 3$ ) can be taken independent of  $n \geq 1$ , since  $\|u_{\tau n}\|_H$  is uniformly bounded in  $n$ . So we have by the lower semicontinuity of  $\varphi^t$  that

$$\varphi^t(u(t)) \leq \liminf_{n \rightarrow \infty} \varphi^t(u_n(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta)) < \infty \quad (3.55)$$

for all  $t \in [\tau + \delta, T]$ . Since  $\delta > 0$  is arbitrary, this estimate implies that  $u(t) \in D(\varphi^t)$  for  $t \in (\tau, T]$ . Here, for any  $\delta \in (0, T - \tau)$ , we consider the following problem:

$$\begin{cases} \frac{dv}{dt} + \partial \varphi^t(v(t)) + B^t(v(t)) + R(t)v(t) \ni f(t) & \text{for } t \in [\tau + \delta, T], \\ v(\tau + \delta) = u(\tau + \delta) \in D(\varphi^{\tau+\delta}). \end{cases} \quad (3.56)$$

The problem (3.56) has a unique strong solution and we have

$$\|u_n(t) - v(t)\|_H^2 \leq \|u_n(\tau + \delta) - v(\tau + \delta)\|_H^2 \exp \left\{ \int_{\tau+\delta}^t 2 \left( \frac{C_6^2}{kC_1} \|v(s)\|_V^2 + \beta_8 \right) ds \right\}$$

for every  $t \in [\tau + \delta, T]$ . Letting  $n \rightarrow \infty$ , then we have  $\|u_n(\tau + \delta) - v(\tau + \delta)\|_H \rightarrow \|u(\tau + \delta) - v(\tau + \delta)\|_H = 0$ , and consequently  $\|u_n(t) - v(t)\|_H \rightarrow 0$  for all  $t \in [\tau + \delta, T]$ . Hence we obtain that  $\|u(t) - v(t)\|_H = 0$  for all  $t \in [\tau + \delta, T]$ , and  $u(t)$  is absolutely continuous on  $[\tau + \delta, T]$ . Since  $\delta > 0$  is arbitrary and  $u \in C([\tau, T]; H)$ , we see  $u$  is a strong solution on  $[\tau, T]$  with  $u(\tau) = u_\tau$ . It is easy to see that  $u$  depends continuously on  $u_\tau$  in  $H$ . Making use of (3.55), noting  $\delta, T > 0$  are arbitrary, we can show that  $u$  is bounded on  $[\delta, \infty)$  with respect to  $V$ -norm. Thus we have established the theorem.

#### PROOF OF THEOREM 2.4.

Since  $\partial\varphi^t + R(t)$  is semi-coercive to (AE), we obtain an a priori estimate by Lemma 3.8, that is,  $\|u(t)\|_H^2 \leq A_0 + A'_0 \|u(\tau)\|_H^2$  for all  $t \geq \tau$ , where  $A_0 = \frac{1}{\gamma'} \|f\|_{L^2(-\infty, \infty; H)}^2 + \frac{2C'}{\gamma'}$ ,  $A'_0 = 1$  if  $f \in L^2(-\infty, \infty; H)$ , and  $A_0 = \frac{1}{(\gamma')^2} \|f\|_{L^\infty(-\infty, \infty; H)}^2 + \frac{2C'}{\gamma'}$ ,  $A'_0 = 1$  if  $f \in L^\infty(-\infty, \infty; H)$  (we note  $C'$  depends on  $\|u(\tau)\|_H$ ).

Let  $u_\tau \in D(\varphi^\tau)$ . Due to Theorem 2.2, there exists a local solution  $u$  of (AE) on some  $[\tau, \tau + \tau_0]$ , and  $\varphi^t(u(t))$  is absolutely continuous on  $[\tau, \tau + \tau_0]$  by Theorem 2.2. By the same way as was used in the proof of Theorem 2.3, for an arbitrary given  $\delta > 0$  we have  $\varphi^t(u(t)) \leq (a_2(\delta)/\delta + a_3(\delta)) \exp(a_1(\delta))$  for any  $t \in [\tau + \delta, \tau + \tau_0]$ . Though  $C'$  depends on  $\|u(\tau)\|_H$ , but  $A_0$  and consequently  $a_i(\delta)$  are independent of any finite interval  $[\tau + \eta, T]$ , therefore we can obtain a global solution  $u$  on  $[\tau, \infty)$  with  $u(\tau) = u_\tau$ .

As for  $u_\tau \in \overline{D(\varphi^\tau)}^H$ , we can employ similar argument to that of the proof of Theorem 2.3. Indeed, let  $\{u_{\tau_n}\} \subset D(\varphi^\tau)$  be a sequence such that  $\|u_{\tau_n} - u_\tau\|_H \rightarrow 0$  as  $n \rightarrow \infty$  and let  $u_n(t)$  be a strong solution on  $[\tau, T]$  with  $u(\tau) = u_\tau$ . Then it holds that  $\|u_n(t)\|_H^2 \leq A_0 + A'_0 \|u_{\tau_n}\|_H^2$  for all  $t \geq \tau$ . Noting that  $\|u_{\tau_n}\|_H$  is uniformly bounded in  $n$ , we find that  $C'$  and  $A_0$  stay in a bounded region. Hence  $a_i(\delta)$  ( $i = 1, 2, 3$ ) can be chosen independent of  $n \geq 1$ . The remaining part of the proof is the repetition of that of Theorem 2.3. So we skip it.

## 4 Applications

We can treat several equations within a framework of §2, for example, the heat conduction equation, the Navier-Stokes equation, the heat convection equation and MHD equation. In this section we deal with the Navier-Stokes equation and the heat convection equation.

Throughout this section, we make some assumptions on  $\Omega(t)$  ( $t \in \mathbb{R}$ ):

(D1) The domain  $\overline{\Omega(t)}$  is included in a two-dimensional ball  $B = B(O, d/2) \subset \mathbb{R}^2$  with the center  $O$  and a radius  $d/2$ .

(D2) The boundary  $\partial\Omega(t) = \Gamma(t)$  is a  $C^3$  class smooth one with respect to both  $x$  and  $t$ . For each  $t$ ,  $\Gamma(t)$  is a simple closed curve in  $\mathbb{R}^2$ . Moreover,  $\sup_{-\infty < t < \infty} |\omega_\varepsilon(t)| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  where

$\omega_\varepsilon(t) = \{x \in B; \text{dist}(x, \Gamma(t)) \leq \varepsilon\}$  and  $|\omega_\varepsilon(t)|$  stands for the volume of  $\omega_\varepsilon(t)$ .

(D3) Let  $\hat{\Omega} = \bigcup_{-\infty < t < \infty} (\Omega(t) \times \{t\})$  and  $\Omega(s, t) = \bigcup_{s < r < t} (\Omega(r) \times \{r\})$ . Then  $\hat{\Omega}$  is covered by at most countable slices  $\Omega(s_i, t_i)$  ( $i \in \mathbb{Z}$ ) such that, for every  $i \in \mathbb{Z}$ ,  $\Omega(s_i, t_i)$  is mapped onto a cylindrical domain  $\Omega(s_i) \times (s_i, t_i)$  by a diffeomorphism  $\Phi_i$  which is of class  $C^3$  up to the boundary and preserve the time coordinate  $t$ .

(D4) Diffeomorphisms  $\Phi_i$  are represented by functions what are uniformly bounded together with those first derivatives with respect to  $x, t$  and  $i$ .

As usual  $H^k(Q)$ ,  $H_0^k(Q)$  are Sobolev spaces and  $H_\sigma(Q)$ ,  $H_\sigma^1(Q)$  are solenoidal spaces. We denote by  $P(Q)$  the orthogonal projection operator from  $L^2(Q)$  onto  $H_\sigma(Q)$ .

#### 4.1 Navier-Stokes equation

In this subsection, we consider the Navier-Stokes equation in a time-dependent bounded domain  $\Omega(t) \subset \mathbb{R}^2$ :

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u + f(t) & \text{in } \hat{\Omega}_\tau, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}_\tau, \end{cases} \quad (4.1)$$

$$u|_{\partial\Omega(t)} = \beta(x, t), \quad \text{for any } t \in (\tau, \infty), \quad (4.2)$$

$$u|_{\tau} = u_\tau \quad \text{in } \Omega(\tau), \quad (4.3)$$

where  $\hat{\Omega}_\tau = \cup_{\tau < t < \infty} (\Omega(t) \times \{t\})$ ,  $\tau \in \mathbb{R}$ . Here  $u = u(x, t)$  means the velocity vector,  $p = p(x, t)$  is the pressure. We put the viscosity  $\nu = 1$  for the sake of simplicity.

In addition to (D1)~(D4), we make an assumption:

(S1) The function  $\beta(x, t)$  is a boundary value of a solenoidal smooth function  $b(x, t) = \operatorname{rot} c(x, t)$  ( $c \in C^3$ ) such that  $\int_{\Gamma(t)} \beta \cdot n d\sigma = 0$ , where  $n$  is an outer normal vector.

The following lemma play an important role:

**Lemma 4.1** (Temam<sup>13</sup>p.470, Temam<sup>14</sup>p.118) *Let  $\beta \in H^{\frac{3}{2}}(\Gamma(t))$  such that  $\int_{\Gamma(t)} \beta \cdot n d\sigma = 0$ .*

*Then for any  $\varepsilon > 0$ , there exists a function  $b = b_\varepsilon(x, t)$  on  $B$  such that  $b_\varepsilon \in H^2(B)$ ,  $\operatorname{div} b_\varepsilon = 0$  in  $B$ ,  $b_\varepsilon = \beta$  on  $\partial\Omega(t)$  and  $|((u \cdot \nabla)b_\varepsilon(t), u)| \leq \varepsilon \|\nabla u\|_{L^2(\Omega(t))}^2$  for all  $u \in H_\sigma^1(\Omega(t))$ , where  $(\cdot, \cdot)$  is the inner product in  $L^2(\Omega(t))$ .*

Here we make changing of variable to transform the equation to a homogeneous boundary value problem. Using  $b = b_\varepsilon$ , we put  $u = u^* + b$ . We abbreviate  $*$  and using same letter  $u$ . Let  $\tau \in \mathbb{R}$ , then we have:

$$\begin{cases} u_t + (u \cdot \nabla)u &= -\nabla p + \Delta u - (u \cdot \nabla)b - (b \cdot \nabla)u \\ &\quad -b_t - (b \cdot \nabla)b + \Delta b + f(t) \quad \text{in } \hat{\Omega}_\tau, \\ \operatorname{div} u &= 0 \quad \text{in } \hat{\Omega}_\tau, \\ u|_{\partial\Omega(t)} &= 0 \quad \text{for any } t \in (\tau, \infty), \quad u(\tau) = u_\tau - b(\cdot, \tau) \quad \text{in } \Omega(\tau). \end{cases}$$

For the Navier-Stokes equation (see the work<sup>12</sup>), we introduce the following convex function  $\varphi^t$  and its subdifferential  $\partial\varphi^t$ :

$$\varphi_B(u) = \begin{cases} \frac{1}{2} \int_B |\nabla u|^2 dx, & u \in H_\sigma^1(B), \\ +\infty, & u \in H_\sigma(B) \setminus H_\sigma^1(B), \end{cases}$$

$$I_{K(t)}(u) = \begin{cases} 0, & u \in K(t), \\ +\infty, & u \in H_\sigma(B) \setminus K(t), \end{cases}$$

where  $K(t) = \{u \in H_\sigma(B) ; u = 0 \text{ a.e. } B \setminus \Omega(t)\}$ . We put

$$\varphi^t(u) = \varphi_B(u) + I_{K(t)}(u) \quad \text{for } u \in H_\sigma(B).$$

The effective domain  $D(\varphi^t)$  and the domain  $D(\partial\varphi^t)$  of the subdifferential  $\partial\varphi^t$  are as follows:

$$\begin{aligned} D(\varphi^t) &= \{u \in H_\sigma(B) : u|_{\Omega(t)} \in H_\sigma^1(\Omega(t)), u|_{B \setminus \Omega(t)} = 0\}, \\ D(\partial\varphi^t) &= \{u \in H^2(\Omega(t)) \cap H_\sigma^1(\Omega(t)), u = 0 \text{ a.e. } B \setminus \Omega(t)\}, \\ \partial\varphi^t(u) &= \{g \in H_\sigma(B) : P(\Omega(t))g|_{\Omega(t)} = -P(\Omega(t))\Delta u|_{\Omega(t)}\}. \end{aligned}$$

Now, putting  $f$  is 0 outside  $\Omega(t)$  and using the same symbol, we have the following abstract Navier-Stokes equation (ANS):

$$(ANS) \quad \frac{du}{dt} + \partial\varphi^t(u) + B^t(u(t), u(t)) + R(t)u(t) \ni P(B)\tilde{f}(t), \quad t \in (\tau, \infty), \quad (4.4)$$

$$u(\tau) = u_\tau - b(\cdot, \tau), \quad \tau \in \mathbb{R} \quad (4.5)$$

where  $B^t(u(t), u(t)) = P(B)(u \cdot \nabla)u$ ,  $R(t)u(t) = P(B)((u \cdot \nabla)b + (b \cdot \nabla)u)$  and  $\tilde{f}(t) = -b_t - (b \cdot \nabla)b + \Delta b + f(t)$ .

Assumptions (A1), (A2), (A3) and (A5) are satisfied.

Now we review a lemma in Ōeda<sup>10)</sup> and by which we verify (A4) below.

**Lemma 4.2** *Assume (D1)  $\sim$  (D4). There exist constants  $C_2 > 0, C_3$  and  $\tau_0 > 0$  such that for any  $t_0 \in (-\infty, \infty)$  and  $v_0 \in D(\varphi^{t_0})$  we can take numbers  $s^*(t_0), t^*(t_0)$  and an  $L^2(B)$ -valued function  $v(\cdot)$  on a closed interval  $I(t_0) \equiv [\max\{t_0 - \tau_0, s^*\}, \min\{t_0 + \tau_0, t^*\}]$  satisfying*

$$\|v(t) - v_0\|_{L^2(B)} \leq C_2 \cdot |t - t_0| \cdot \varphi^{t_0}(v_0)^{1/2} \quad \text{for every } t \in I(t_0), \quad (4.6)$$

$$\varphi^t(v(t)) \leq \varphi^{t_0}(v_0) + C_3 \cdot |t - t_0| \cdot \varphi^{t_0}(v_0) \quad \text{for every } t \in I(t_0). \quad (4.7)$$

Proof of Lemma 4.2. By (D3), for any  $t_0 \in (-\infty, \infty)$  there exists  $J_i = (s_i, t_i)$  such that  $t_0 \in J_i$ . Due to Lemma 3.2 of Ōtani-Yamada<sup>12)</sup> and (D3), we see for any interval  $J_i$ , there exist constants  $C_{i2} > 0, C_{i3} > 0$  (which depends on  $J_i$ ) and  $\tau_0 > 0$  such that for every  $t_0 \in J_i$  and  $v_0 \in D(\varphi^{t_0})$  there exists an  $H(B)$ -valued function  $v(\cdot)$  satisfying estimates (4.6) and (4.7) by replacing  $C_2, C_3$  with  $C_{i2}, C_{i3}$ . Thanks to (D4), we find  $C_2, C_3$  such that  $\sup_{i \in \mathbb{Z}} C_{i2} \leq C_2 < +\infty$  and  $\sup_{i \in \mathbb{Z}} C_{i3} \leq C_3 < +\infty$  hold. By (D4)  $\tau_0 > 0$  can be taken uniformly in  $J_i$ . For  $t_0$ , putting  $s^*(t_0) = s_i, t^*(t_0) = t_i$ , we get the lemma. Hence (A4) holds.

We note that  $D(\varphi^\tau) = H_\sigma^1(\Omega(\tau)) \subset V = H_\sigma^1(B) \subset H = H_\sigma(B)$ . Moreover  $B^t, R(t)$  satisfy (B0) $\sim$ (B4), (R1) $\sim$ (R3) respectively.

Here we find that  $\partial\varphi^t + R(t)$  is coercive. In fact, by means of (B0) and Lemma 4.1 (say we take  $\varepsilon = \frac{1}{2}$ ), if  $g \in \partial\varphi^t(u)$ , then we have  $(g, u)_{L^2(\Omega(t))} = \|\nabla u\|_{L^2(\Omega(t))}^2 \geq \gamma \|u\|_{H_\sigma^1(B)}^2$ , where  $\gamma$  is a domain constant determined by  $B$ .

Then, applying Theorem 2.3, we have:

**Theorem 4.1** *Suppose (D1) $\sim$ (D4) and (S1). Assume  $f \in L^\infty(-\infty, \infty; L^2(B))$  or  $f \in L^2(-\infty, \infty; L^2(B))$ . Then there exists a unique global (in time) strong solution  $u$  on  $[\tau, \infty)$  of (ANS) with  $u(\tau) = u_\tau \in H_\sigma^1(\Omega(\tau))$  and  $u$  is bounded with respect to  $H_\sigma^1(B)$ -norm.*

## 4.2 Heat convection equation

We consider the heat convection equation of Boussinesq approximation in a time-dependent bounded domain  $\Omega(t) \subset \mathbb{R}^2$  (see literatures<sup>6),9)</sup>):

$$\begin{cases} u_t + (u \cdot \nabla)u = -\rho^{-1}\nabla p + \{1 - \alpha(\theta - \Theta_0)\}g + \nu\Delta u & \text{in } \hat{\Omega}_\tau, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}_\tau, \\ \theta_t + (u \cdot \nabla)\theta = \kappa\Delta\theta & \text{in } \hat{\Omega}_\tau, \end{cases} \quad (4.8)$$

$$u|_{\partial\Omega(t)} = \beta(x, t), \quad \theta|_{\partial\Omega(t)} = \chi(x, t), \quad \text{for any } t \in (\tau, T), \quad (4.9)$$

$$u(\tau) = u_\tau \quad \theta(\tau) = h_\tau \quad \text{in } \Omega(\tau), \quad (4.10)$$

where  $\hat{\Omega}_\tau = \cup_{\tau < t < \infty} (\Omega(t) \times \{t\})$ . Here  $u = u(x, t)$  means the velocity vector,  $p = p(x, t)$  is the pressure and  $\theta = \theta(x, t)$  is the temperature;  $\nu, \kappa, \alpha, \rho$  are physical constants what mean the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion and the density at  $\theta = \Theta_0$  respectively, and  $g = g(x)$  stands for the gravitational vector.

In addition to (D1)~(D4) and (S1), we assume:

(S2) Put  $\Theta_0 > 0$ .  $\chi = \chi(x, t)$  is a smooth function of class  $C^2$  and  $0 \leq \chi \leq \Theta_0$  ( $-\infty < t < \infty$ ). To transform the equation to the problem with the homogenous Dirichlet boundary condition, we need lemmas. Lemma 4.1 of the preceding subsection is available.

Concerning  $\theta(t)$ , we state the next lemma as an n-dimensional case.

**Lemma 4.3** *Let  $\tau \in \mathbb{R}$  be any given. Let  $\Omega(t) \subset \mathbb{R}_x^n$  and let  $\Gamma(t)$  be a simple closed curve or surface. Then, for a function  $\chi(x, t)$  with the property (S2), there is a function  $\bar{\theta}(x, t)$  which satisfies the following (i) ~ (iv): (i)  $\bar{\theta}(x, t) = \chi(x, t)$  on  $\hat{\Gamma}_\tau = \cup_{\tau < t < \infty} (\Gamma(t) \times \{t\})$ . (ii)  $\bar{\theta}(x, t) \in C_0^2(\mathbb{R}_x^n)$  for any fixed  $t$  and  $\bar{\theta}, \bar{\theta}_t$  are continuous for  $t \geq \tau$ . Furthermore  $0 \leq \bar{\theta}(x, t) \leq \Theta_0$  for  $x \in \Omega(t), t \geq \tau$ . (iii) For any  $\varepsilon > 0$  and  $p > 1$ , we can retake  $\bar{\theta}$ , if necessary, such that  $\sup_{t \geq \tau} \|\bar{\theta}(t)\|_{L^p} < \varepsilon$ . (iv) For an arbitrary  $\varepsilon > 0$ , there exists  $\bar{\theta}$  such that  $\|(u \cdot \nabla)\bar{\theta}(t)\|_{L^2} \leq \varepsilon \|\nabla u\|_{L^2}$  for all  $u \in H_0^1(\Omega(t))$  and all  $t \geq \tau$ , provided we reconstruct  $\bar{\theta}$ , if necessary.*

We omit the proof. See the literature<sup>9)</sup>. Here we make changing of variables to transform the system of equations to a homogeneous and nondimensional one. We put  $(x, y) = d(x^*, y^*), t = \frac{d^2}{\nu}t^*, u = \hat{u} + b, \theta = \hat{\theta} + \bar{\theta}, \hat{u} = \frac{\nu}{d}u^*, \hat{\theta} = \frac{\nu\Theta_0}{\kappa}\theta^*, p = \frac{\rho\nu^2}{d^2}p^*$ . Abbreviating asterisks  $*$  and using the same letters, then we have:

$$\begin{aligned} u_t + (u \cdot \nabla)u &= -\nabla p + \Delta u - (u \cdot \nabla)b - (b \cdot \nabla)u - R_a\theta \\ &\quad - b_t - (b \cdot \nabla)b + \Delta b + d^3g/\nu^2 - R_a(\bar{\theta} - P^{-1}) \quad \text{in } \hat{\Omega}_\tau, \\ \operatorname{div} u &= 0 \quad \text{in } \hat{\Omega}_\tau, \\ \theta_t + (u \cdot \nabla)\theta &= P^{-1}\Delta\theta + P^{-1}\Delta\bar{\theta} - \bar{\theta}_t - (u \cdot \nabla)\bar{\theta} - (b \cdot \nabla)\theta - (b \cdot \nabla)\bar{\theta} \quad \text{in } \hat{\Omega}_\tau, \\ u|_{\partial\Omega(t)} &= 0, \quad \theta|_{\partial\Omega(t)} = 0 \quad \text{for any } t \in (\tau, T), \\ u(\tau) &= u_\tau - b(\cdot, \tau), \quad \theta(\tau) = h_\tau - \bar{\theta}(\cdot, \tau) \quad \text{in } \Omega(\tau), \end{aligned}$$

where  $R_a = \alpha g \Theta_0 d^3 / \kappa \nu$  (Rayleigh type) and  $P = \nu / \kappa$  (Prandtl number).

Here we introduce a proper lower semicontinuous function on  $U = (u, \theta)$  (see the work<sup>6)</sup>):

$$\varphi_B(U) = \begin{cases} \frac{1}{2} \int_B (|\nabla u|^2 + \frac{\kappa}{\nu} |\nabla \theta|^2) dx, & U \in H_\sigma^1(B) \times H_0^1(B), \\ +\infty, & U \in (H_\sigma(B) \times L^2(B) \setminus (H_\sigma^1(B) \times H_0^1(B))), \end{cases}$$

$$I_{K(t)}(U) = \begin{cases} 0, & U \in K(t), \\ +\infty, & U \in (H_\sigma(B) \times L^2(B)) \setminus K(t), \end{cases}$$

where  $K(t) = \{U \in H_\sigma(B) \times L^2(B); U = 0 \text{ a.e. } B \setminus \Omega(t)\}$ . Then we put

$$\varphi^t(U) = \varphi_B(U) + I_{K(t)}(U).$$

The effective domain  $D(\varphi^t)$  of  $\varphi^t$  is as follows:

$$D(\varphi^t) = \{U \in H_\sigma(B) \times L^2(B); U|_{\Omega(t)} \in H_\sigma^1(\Omega(t)) \times H_0^1(\Omega(t)), U|_{B \setminus \Omega(t)} = 0\}.$$

Moreover we consider a subdifferential operator  $\partial\varphi^t$ .

$$\begin{aligned} D(\partial\varphi^t) &= \{U \in H_\sigma(B) \times L^2(B) ; (H^2(\Omega(t)) \cap H_\sigma^1(\Omega(t))) \\ &\quad \times (H^2(\Omega(t)) \cap H_0^1(\Omega(t))), U = 0 \text{ a.e. } B \setminus \Omega(t)\}, \\ \partial\varphi^t(U) &= \{f \in H_\sigma(B) \times L^2(B) ; P(\Omega(t))f|_{\Omega(t)} = A(\Omega(t))U|_{\Omega(t)}\}, \end{aligned}$$

where we used symbols  $A(Q) = (-P_\sigma(Q)\Delta, -\frac{\kappa}{\nu}\Delta)$ ,  $P(Q) = (P_\sigma(Q), 1_Q)$ , and  $P_\sigma(Q)$  is a projection, i.e.,  $P_\sigma(Q); L^2(Q) \rightarrow H_\sigma(Q)$ .

Now we define the abstract heat convection equation (AHC) as follows:

$$(AHC) \quad \frac{dU}{dt} + \partial\varphi^t(U) + B^t(U(t), U(t)) + R(t)U(t) \ni P(B)f(t), \quad t \geq \tau, \quad (4.11)$$

$$U(\tau) = (u_\tau - b(\cdot, \tau), h_\tau - \bar{\theta}(\cdot, \tau)), \quad \tau \in \mathbb{R}, \quad (4.12)$$

$$\begin{aligned} B^t(U(t), U(t)) &= (P_\sigma(B)(u \cdot \nabla)u, (u \cdot \nabla)\theta), \\ R(t)U(t) &= (P_\sigma(B)((u \cdot \nabla)b + (b \cdot \nabla)u + R_a\theta), (u \cdot \nabla)\bar{\theta} + (b \cdot \nabla)\theta), \\ f(t) &= (-b_t - (b \cdot \nabla)b + \Delta b + \frac{d^3 g}{\nu^2} - R_a(\bar{\theta} - \frac{\kappa}{\nu}), -\bar{\theta}_t + P^{-1}\Delta\bar{\theta} - (b \cdot \nabla)\bar{\theta}). \end{aligned}$$

Assumptions (A1)~(A5) are satisfied by (D1)~(D4). We note that  $D(\varphi^\tau) = H_\sigma^1(\Omega(\tau)) \times H_0^1(\Omega(\tau)) \subset V = H_\sigma^1(B) \times H_0^1(B) \subset H = H_\sigma(B) \times L^2(B)$ . Operators  $B^t$  and  $R(t)$  satisfy (B0)~(B4) and (R1)~(R3) respectively.

To show the semi-coerciveness to (AHC), we need two lemmas (see works<sup>1),9</sup>). First we state a variant of the maximum principle.

**Lemma 4.4** *Let  $U = (u, \theta)$  be a strong solution on  $[\tau, T]$ . Then there exist functions  $\theta_1$  and  $\theta_2$  such that the following properties are satisfied:*

$$\theta(\cdot, t) = \theta_1(\cdot, t) + \theta_2(\cdot, t), \quad -\frac{\kappa}{\nu} \leq \theta_1(\cdot, t) \leq \frac{\kappa}{\nu}, \quad (4.13)$$

$$\|\theta_2(t)\|_{L^2(B)} \leq \{ \|(\theta - \frac{\kappa}{\nu})_+(\tau)\|_{L^2(B)} + \|\theta_-(\tau)\|_{L^2(B)} \} \exp(-2\kappa(t - \tau)/\nu), \quad (4.14)$$

where  $f_+ = \max\{f, 0\}$ ,  $f_- = \max\{-f, 0\}$ .

Next, an a priori estimate on  $\theta$  is as follows.

**Lemma 4.5** *Let  $U = (u, \theta)$  be a strong solution on  $[\tau, T]$ . Then we have*

$$\|\theta(t)\|_{L^2(B)} \leq \frac{\kappa}{\nu} |B|^{\frac{1}{2}} + \|\theta(\tau)\|_{L^2(B)} \exp(-2\kappa(t - \tau)/\nu) \quad \text{for } t \in [\tau, T], \quad (4.15)$$

where  $|B|$  is the volume of  $B$ .

By the influence of  $\theta(t)$ ,  $\partial\varphi^t + R(t)$  of (AHC) is not coercive, but we have

**Lemma 4.6**  *$\partial\varphi^t + R(t)$  is semi-coercive to (AHC).*

PROOF OF LEMMA 4.6.

Let  $U = (u, \theta)$ ,  $G(t) \in \partial\varphi^t(U(t))$ . We use symbols  $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega(t)) \times L^2(\Omega(t))}$  and  $\|\cdot\| = \|\cdot\|_{L^2(\Omega(t))}$ . In (AHC), we remember

$$R(t)U(t) = (P_\sigma(B)((u \cdot \nabla)b + (b \cdot \nabla)u + R_a\theta), (u \cdot \nabla)\bar{\theta} + (b \cdot \nabla)\theta).$$

Let  $\tau \in \mathbb{R}$  be arbitrarily fixed and  $t \geq \tau$ . Recalling  $(P_\sigma(B)(b \cdot \nabla)u, u) = 0$ ,  $((b \cdot \nabla)\theta, \theta) = 0$ , making use of Lemma 4.1, Lemma 4.3 and Lemma 4.5, we calculate the following:

$$\begin{aligned} & (G(t), U(t)) + (R(t)U(t), U(t)) \\ &= \|\nabla u(t)\|^2 + \frac{\kappa}{\nu} \|\nabla \theta(t)\|^2 + (P_\sigma(B)(u \cdot \nabla)b, u) \\ &+ (P_\sigma(B)(b \cdot \nabla)u, u) + (P_\sigma(B)R_a\theta, u) + ((u \cdot \nabla)\bar{\theta}, \theta) + ((b \cdot \nabla)\theta, \theta) \\ &\geq \|\nabla u(t)\|^2 + \frac{\kappa}{\nu} \|\nabla \theta(t)\|^2 - \frac{1}{4} \|\nabla u(t)\|^2 - \frac{|R_a|^2}{2C_1} \|\theta(t)\|^2 - \frac{1}{4} \|\nabla u(t)\|^2 \\ &- \frac{\kappa}{4\nu} \|\nabla \theta(t)\|^2 - \frac{1}{4} \|\nabla u(t)\|^2 - \frac{|R_a|^2}{C_1} \|\theta(t)\|^2 \\ &\geq \frac{1}{4} \|\nabla u(t)\|^2 + \frac{3\kappa}{4\nu} \|\nabla \theta(t)\|^2 - \frac{2|R_a|^2}{C_1} \|\theta(t)\|^2 \\ &\geq \frac{1}{4} \|\nabla u(t)\|^2 + \frac{3\kappa}{4\nu} \|\nabla \theta(t)\|^2 - \frac{2|R_a|^2}{C_1} \left( \frac{2\kappa^2}{\nu^2} |B| + 2\|\theta(\tau)\|_{L^2(B)}^2 \right) \\ &\geq \frac{1}{C(\Omega(t)) + 1} \left( \frac{1}{4} \|\nabla u(t)\|^2 + \frac{3\kappa}{4\nu} \|\nabla \theta(t)\|^2 \right) - \frac{2|R_a|^2}{C_1} \left( \frac{2\kappa^2}{\nu^2} |B| + 2\|\theta(\tau)\|_{L^2(B)}^2 \right) \\ &\geq \frac{1}{C(B) + 1} \cdot \min\left\{ \frac{1}{4}, \frac{3\kappa}{4\nu} \right\} \|U(t)\|_{H_\sigma^1(B) \times H_0^1(B)}^2 - C', \end{aligned}$$

where  $C(\Omega(t))$ ,  $C(B)$  are domain constants and we know  $C(\Omega(t)) \leq C(B)$ ;  $C' = \frac{2|R_a|^2}{C_1} \left( \frac{2\kappa^2}{\nu^2} |B| + 2\|\theta(\tau)\|_{L^2(B)}^2 \right)$ . Symbols  $|R_a|$  and  $|B|$  stand for the magnitude of  $R_a$  and the volume of  $B$  respectively. Hence  $\partial\varphi^t + R(t)$  is semi-coercive to (AHC). We proved the lemma.

Applying Theorem 2.4, we have:

**Theorem 4.2** *Suppose (D1)~(D4), (S1) and (S2). Assume  $f \in L^\infty(-\infty, \infty; L^2(B) \times L^2(B))$  (note  $f \notin L^2(-\infty, \infty; L^2(B) \times L^2(B))$ ). Then there exists a unique global (in time) strong solution  $(u, \theta)$  on  $[\tau, \infty)$  of (AHC) with  $(u(\tau), \theta(\tau)) = (u_\tau, \theta_\tau) \in H_\sigma^1(\Omega(\tau)) \times H_0^1(\Omega(\tau))$ . Moreover  $u$  and  $\theta$  are bounded with respect to  $H_\sigma^1(B)$ -norm and  $H_0^1(B)$ -norm respectively.*



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# 時間依存定義域をもつ劣微分作用素により生成された 発展方程式

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**要旨：**現象を考察する領域が時間とともに変化する問題を扱う．このような問題は時間依存定義域を伴う発展方程式をもたらす．本論文では当該方程式が或る条件下で有界な時間的大域解を有することを示す．

**キーワード：**劣微分作用素, 時間依存定義域

